THE BONEYARD BOOK

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(Rubel) Is the ring (or algebra) of entire functions of order ρ isomorphic to that of order ρ' if $\rho \neq \rho'$?

Answer is <u>no</u> by E.G. Straus.

~~(Remark) [illegible] is a completely [illegible] group with [illegible]~~

~~Prüfer's theorem says that G is a direct sum of cyclic groups~~

~~Is there a point-set topologized proof of this using basic subgroups?~~

(Rotman). If A is an abelian group, set $A^* = \mathrm{Hom}(A, Z)$, Z = integers

Is $A^* \cong A^{***}$?

(Granirer): If a group contains a free subsemigroup with two generators. Does it necessarily contain also a free subgroup with two generators?

This question is connected with the following question on amenable semigroups: If G is an amenable group, then is every subsemigroup also amenable?

A.H. Frey proved that a group does not contain a free subsemigroup with two generators if and only in every subsemigroup any two right ideals have nonvoid intersection. And also: If a left amenable group does not contain free subsemigroup with two generators then every subsemigroup is left amenable.

Let $G = \{a, b; : ab^{-1}ab^{-1} = e\}$ The semigroup on a, b is free Appel-Djorup

(Rubel) Let B be a vector subspace (B is not assumed to be closed) of $L^2(-\infty, \infty)$ such that if $f \varepsilon B$ and $|g(x)| \leq |f(x)|$ a.e. then $g \varepsilon B$, and such that if ~~[illegible]~~ $f \varepsilon B$ then $f^\wedge \varepsilon B$, where $f^\wedge$ is the Fourier transform of f. Is it true that either $B = L^2$ or $B = \{0\}$?

~~Solution: — No. Let $P[-\infty, \infty]$ be the class of finite linear combinations of positive definite functions on~~

(E C Weinberg)

Can every torsion-free abelian group be provided with an archimedean lattice-order?

This will not be the case if there exists a torsion-free abelian group of cardinality greater than c with the property that every pure subgroup is indecomposable.

There are none

No. Complete solution to be published by Griffith in Proc AMS.

(Wetzel) Let $\mathcal{F}$ be a family of functions f holomorphic in the unit disc, such that for each z in the disc,

$$\{ f(z) : f \in \mathcal{F} \} \text{ is countable.}$$

Is the family $\mathcal{F}$ countable?

Dixon ~~has a short~~ proof that the continuum hypothesis is true if there exists an ~~uncountable family~~.
He shows that if continuum hypothesis is false, then each family is countable.

If $c = \aleph_1$ such a family can have power c (Erdős)
The following problem remains: Assume that for every z $\{ f(z)\ f \in \mathcal{F} \}$ has power $< c$. Is the family $\mathcal{F}$ of power $< c$? If $c = \aleph_1$ this is false. What happens if $c > \aleph_1$?

~~It might be to someone's benefit to consider the situation for a UF domain which~~ properly contains an $\cong$ copy of ~~the ring of rational integers~~ !?

(Erdős and Heilbronn) Let $a_1, a_2, \ldots, a_k$ be k distinct residues (mod p). Is it true that the number of distinct residues of the form $a_i + a_j$, $i \neq j$, is at least $\min(2k-3, p)$?

(Erdős) Are there for every $\varepsilon > 0$ more than $n(1-\varepsilon)$ integers $1 < a_1 < a_2 < \dots < a_k \le n$, $k > n(1-\varepsilon)$ so that if $a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_r} = a_{j_1} \cdot a_{j_2} \cdot \ldots \cdot a_{j_l}$, $i_1 < i_2 < \dots < i_r$; $j_1 < j_2 < \dots < j_l$ then $r = l$?
e.g. the numbers $2(2m+1)$ have this property, but their number is only $n/4$.

Selfridge showed that if n is suff large one can give $\frac{n}{e}(1-\varepsilon)$ such integers.

(Erdös) Let $a_i \pmod{n_i}$, $1 \le i \le k$ be such that there is an integer u for which $u \not\equiv a_i \pmod{n_i}$, $1 \le i \le k$. Then this integer can be chosen to satisfy $0 < u \le 2^k$. This sharpens a conjecture of Stein. He assumes $n_1 < \ldots < n_k$ and that no x satisfies two of the congruences (it then follows that there is a u which does not satisfy any of the congruences $a_i \pmod{n_i}$). I can then prove that there is a $0 < u \le k 2^k$ for which $u \not\equiv a_i \pmod{n_i}$.

I have proved Stein's conjecture and collected \$10 from Erdös. The best I can prove for Erdös's conjecture is $0 < u < e^k$.

Selfridge

(Hellerstein-Rubel) Given the field Φ of all meromorphic functions, can the subring E of all entire functions be characterized by algebraic properties?

(Rubel-Shields) Consider the lune $\Lambda = \{z : |z| < 1,\ |z - \frac{1}{4}| > \frac{3}{4}\}$

If f is bounded and holomorphic (analytic) in Λ, do there exist polynomials p_n and q_n such that

i) $|p_n(z) + q_n(\frac{1}{z})| \leq M$ where $M < \infty$

ii) $p_n(z) + q_n(\frac{1}{z})$ converges pointwise in Λ to f?

(D.S. Kahn). Trivially, $D[X] \cong D'[X] \Longrightarrow D \cong D'$ if D and D' are field. How about if D, D' are integral domains, or even rings?

5. Feb 1970. (Enoch and others at U. of Kentucky have considered this problem. $D \cong D'$ if D is generated by its units.)

12. Aug. 70. Abhyankar-Heinzer. Notices AMS 17, 763 (1970).
Let A be a 1-dimensional affine domain $/k$, k arbitrary field. If B is a ring, $x_1, \dots, x_n$ indeterminants over A, $y_1, \dots, y_n$ indeterminants over B and if $\varphi : A[x_1, \dots, x_n] \to B[y_1, \dots, y_n]$ is an isomorphism, then A is isomorphic to B. If A is not $K[t]$ some $K \supseteq k$, then $\varphi A = B$.

14. Dec 73. Hochster (PAMS), has found commutative D and D' which are not isomorphic, but whose polynomial rings are.
Fossum

(Peressini) Is every closed convex bounded set of constant width in a real Hilbert space a sphere?

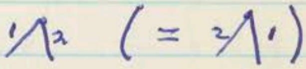
Answer: <u>No</u> by N. T. Hamilton

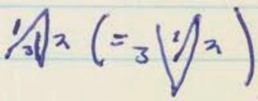
~~(28 August 1963)~~

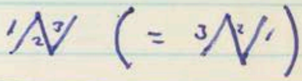
~~Shown: unsolvable even for "reduced" models of ETA in which nothing equivalent to 1 appears, e.g. $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n} \cdots$ is not reduced eventhough $\alpha_i > 0$ for only finitely many i, since $p^{\alpha_k} = 1$ for some k.~~

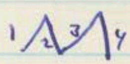
~~Clearly if it is unsolvable for elementary NT it is unsolvable for algebraic NT, i.e., UFD's, in g~~

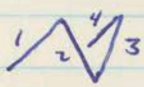
Proposed by Martin Gardner: In how many ways can n stamps be folded? The stamps are assumed to be in a long coil, ~~or~~ not in a rectangular or irregular array.

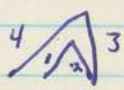
2 stamps: 1 way  12

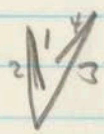
3 stamps 2 ways  312

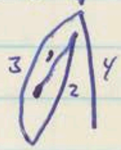
 123

4 stamps 5 ~~4~~ ways  1234

 1243

 4123

 2143

 3124

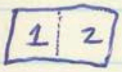
5 stamps around 25 ways

See *Scientific American* August, '63 for further details.

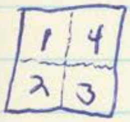
Sci Am Sept '63 gives:

In how many distinct ways can an n by m map be folded into a 1 × 1 square?

1 × 1 — 1 way — $\frac{1}{\ }$



1 × 2 — 1 way — i.e. $\frac{1}{2} = \frac{2}{1}$



2 × 2 — 4 ways

All other arrangements are either impossible to attain or reverses of the above figures. Note that the problem for the 1 × m is the same as the preceding problem.

~~Let α^* be the distribution of fixed points of ψ^*, i.e., $\alpha^*(n) = \text{card}\{a \in N : a \le n,\ \psi^*(a) = a\}$, $n \in N$.~~

~~(3) Prove $\alpha^*(x+y) \le \alpha^*(x) + \alpha^*(y)$~~

~~(4) Relate this to $\pi(x+y) \le \pi(x) + \pi(y)$.~~

Cipner - Erdős. Let $m > \aleph_0$. Is the unit sphere of ~~the unit sphere in~~ an m dimensional Hilbert space the union of fewer than m sets of diameter $< 2-\varepsilon$?

(P Erdös)

Let $a_1 < a_2 < \cdots$ be an infinite sequence of positive density. Denote by $f(x)$ the number of $a_i | a_j$, $a_j \leq x$. Is it true that $\lim f(x)/x = \infty$? ($\overline{\lim}\, f(x)/x = \infty$?)

It can be shown rather easily that if the a sequence has positive upper logarithmic density, then $\overline{\lim}\, f(x)/x = \infty$. Thus there is an affirmative answer to the second question under hypotheses much weaker than the existence of natural density.

However, a sequence may be constructed which has logarithmic density 1 and $\underline{\lim}\, f(x)/x = 0$. Thus the log density approach seems inadaquate to answer the first question. J.R.A

P Erdös

Let $n_1 < n_2 < \dots$ be an infinite sequence of integers. Is it true that

$$\sum_{k=1}^{\infty} \frac{1}{2^{n_k}-1} \text{ is irrational?}$$

Is the same true for $\sum_{n=2}^{\infty} \frac{1}{n!-1}$?

Let $n_k/k \to \infty$. Prove $\sum_{k=1}^{\infty} \frac{n_k}{2^{n_k}}$ is irrational. De Bruijn + I showed this if $n_k > c\, k\,(\log k \log\log k)^{1/2}$.

P Erdős - A Hajnal

Denote by $m(n)$ the smallest integer so that there are $m(n)$ sets A_i, $1 \le i \le m(n)$ each A_i has n elements so that there is no set S for which $A_i \not\subset S$, $A_i \cap S$ non-empty for all $1 \le i \le m(n)$.

Trivially $m(2) = 3$. Further $m(3) = 7$ $m(4)$ is unknown. W Schmidt showed $m(n) > 2^n (1 + O(\frac{1}{n}))$ and Erdős showed $m(n) < c\, n^2 2^n$.

(see P Erdős Math Tidskrift 1963)

P. Erdös and L. Moser. Let $a_1 < a_2 < \dots < a_k$ be a sequence of integers. Let $f(n: a_1, \dots, a_k)$ be the number of solutions of $a = \sum \epsilon_i a_i$, $\epsilon_i = 0$ or 1. Is it true that $f(n: a_1, \dots, a_k) < c\, 2^k / k^{3/2}$? We can prove $c\, 2^k \left(\frac{\log k}{k}\right)^{3/2}$.

In connection with this, the following question can be raised. Let $0 < a_1 < a_2 < \dots < a_k$. Does there exist a sub-sequence containing ck terms where no term of the sub-sequence is the same (sum?) of any number of other terms of the sub-sequence? Perhaps $c = 1/2$. We can only show $\sqrt{k}$ instead of ck.

NO

P. Erdős & L. Moser If there are n numbers, can we find $\left[\frac{n+1}{2}\right]$ of them so that none is the same (sum?) of two others? Here, we can show $\frac{n}{3}$ instead of $\left[\frac{n+1}{2}\right]$

We feel that this last question should be trivial, and perhaps we are overlooking something obvious.

(Horn)
T.G. MacLaughlin

Let $\mathcal{L}_0$ be the lattice of open subsets of the line (ordinary inclusion, union, and intersection are the operations); similarly, let $\mathcal{L}_C$ = the lattice of closed subsets of the line. Is there a lattice homomorphism $\gamma : \mathcal{L}_0 \to \mathcal{L}_C$ that is onto?

[It is easy to see that there is no isomorphism; $\mathcal{L}_C$ has minimal elements and $\mathcal{L}_0$ does not.]

8/7/66: Horn has informed me that this has been settled <u>in the negative</u> by Mackenzie (U.C. Berkeley), for a wider class of metric spaces than just the line.

TGM

~~(modified Euler-function)~~

~~Clearly ϕ^* is dominated by~~

~~ϕ. What are the number-~~

~~theoretic properties of ϕ^*?~~

~~After defining a submosaic in the obvious~~

~~way, prove that $F(n) = \sum' f(m)$, where~~

~~the summation is over submosaics "reflects"~~ ~~gen. supermult.~~

~~from f to F; i.e., $F(\underline{a}\cdot\underline{b}) \leq F(\underline{a})\cdot F(\underline{b})$, if the mosaics of $\underline{a}$ and $\underline{b}$ have~~

P. T. Bateman. Suppose $g(x)$ is a polynomial with rational integral coefficients. Prove that $g(x) - n$ is irreducible (over the field of rational numbers) for most ~~values~~ integral values of n in some sense. For example $x^4 - n$ is irreducible unless n is a perfect square.

A more general result was proved by Skolem in 1921 (Videnskapsselskapets Skr. I, No. 17, 57 pp.) and a still more general result by Dörge (Math. Ann. 96, 176-182) in 1926. For a simple proof of the assertion of the problem see Dörge, Math. Ann. 95 (1925) 247-256.

~~Anonymous, go home!~~

Lee Supowit — is there a function f on R into R such that $x_0 \in R \longrightarrow \exists\ \delta(x_0) > 0$ such that

$f(x) \geq f(x_0)$ whenever $|x - x_0| < \delta(x_0)$

and range f is not countable.

<u>Remark</u> I.P. Natanson: says there is no such f.

Verified, April 9, 1964. Natanson is right. Supowit

James A Marine

R. S. LUTHAR (L) If A is a set of positive measure, show that there is at least one subset of A which is contained in an interval whose length is equal to the measure of the subset.

R Luthar
20-5-64.

Nowhere dense sets with positive measure show the assertion to be false.

An example please?

R Luthar

CONSTRUCT ON $[0,1]$ A CANTOR-LIKE SET NOWHERE DENSE + CLOSED HAVING MEASURE $= r$, $0 < r < 1$ BY REMOVING INTERVALS OF THE RIGHT TYPE: REF: GOFFMAN - REAL FUNCTIONS OR ROYDEN - REAL ANALYSIS

If $R = \{\text{reals}\}$, consider $R - \bigcup_{n=1}^{\infty} I_n$, where $I_n = (r_n - \frac{1}{2^n}, r_n + \frac{1}{2^n})$ and $\{r_n\}_{n=1}^{\infty}$ are the rationals.

M. Newman

Consider the recurrence

$$x_{k+1} = t\,x_k - d\,x_{k-1}\,, \quad x_0 = a, \quad x_1 = b$$

assume $|t| > 2$. Let m be a fixed integer. ~~[illegible]~~ ~~[illegible]~~
Then apart from certain trivial exceptions (such as $(a, b, m) > 1$) prove that one can always find a k such that $(x_k, m) = 1$.

M. Newman

Let Γ_t be the $t \times t$ unimodular group of rational integral matrices. Let Γ_t^n be the fully invariant subgroup of Γ_t generated by the nth powers of all elements of Γ_t, and $\Gamma_t(n)$ the normal subgroup of Γ_t defined by $A \equiv I \pmod{n}$, $A \in \Gamma_t$. Is it true that for $t > 2$, Γ_t^n is of finite index in Γ_t? Is it true that $\Gamma_t^n \supset \Gamma_t(n)$?

M Seibert

Assume Γ_t is one of simplest forms

$a \times a$, $t = a$

Let $n = 2$

then $(a \times a)_t^n \supset (a \times a)_t(n)$

$(a \times a)_t^2 \supset (a \times a)_t(2)$

$(a \times a)_t (a \times a)_t \supset (a \times a)_t(2)$

if $\supset$ is notation for subset

the answer is no

M. Newman

Compute the permanent ('plus' determinant) of the matrix (ζ^{ij}), ζ a primitive $n^{\underline{th}}$ root of unity. Alternatively, determine the total number of permutations σ of the integers $1, 2, \ldots, n$ such that

$$\sum_{k=1}^{n} k\sigma(k) \equiv 0 \pmod{n}$$

(The problems are equivalent)

M. Newman

Can every doubly stochastic matrix (one with row and column sums all 1) be written as the product of elementary 2×2 d.s. matrices; i.e. of matrices of the form

$$P\left\{\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \dotplus I_{n-2}\right\} Q$$

where P, Q are permutation matrices and $\alpha+\beta=1$?

(L. A. Rubel) Let G be an open set in the complex plane, and let P_0 denote the class of polynomials restricted to G. Recursively define P_n by specifying P_{n+1} as the class of functions on G that are pointwise limits on G of a sequence of functions from P_n. The class P_1 has been characterized [*]. Find a characterization of the class P_n for any positive integer n. Solve the corresponding problem for any ordinal number n. In particular, find a characterization of P_2.

[*] S. N. Mergelyan Certain classes of sets and their applications. Soviet Math 2 (1961) pp 590-593

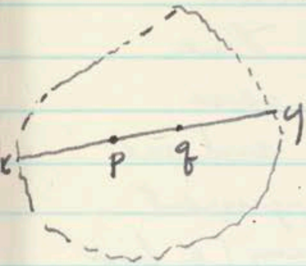
(McLaughlin) By an almost disjoint family (adf) we mean a class of subsets of the natural numbers any two of which have finite intersection. Let $C_0, C_1, \ldots$ be any countable family of subsets of the nat. numbers. Must there exist an uncountable adf $\mathcal{F}$ such that $f_1, f_2 \in \mathcal{F} \Rightarrow$ there is no C_j for which

$$f_1 \subseteq C_j \subseteq \widetabove{f_2 - (f_1 \cap f_2)} \quad ?$$

Such a family always exists

P Erdős 66 II 10

(R.L. Bishop)
1967?

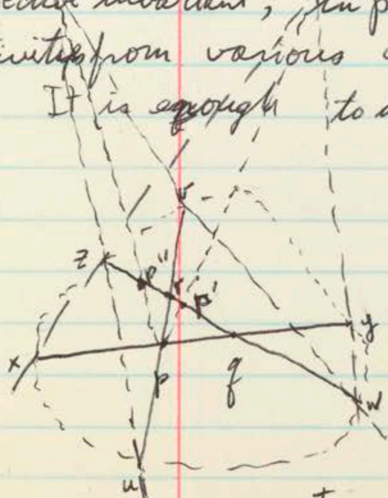


Let S be an open bounded convex set in R^n. For each $p, q \in S$ the line through p, q will intersect the boundary of S in x, y as shown, so the order of the four points is x, p, q, y. Denote the Euclidean distance in R^n by ρ, and define

$$d(p,q) = \left| \log \frac{\rho(q,x)\,\rho(p,y)}{\rho(p,x)\,\rho(q,y)} \right| .$$

Is d a topological metric on S?
(This is known to be true if S is the interior of an ellipse, since then S becomes the hyperbolic plane with metric d.)

R. L. Bishop, 1975. Yes. The property of d in question is the triangle inequality. We use the fact that the cross-ratio is a projective invariant; in particular, it is invariant under perspectivities from various centers. Consider a triangle p, q, r. It is enough to do the plane case.



Let x, y be the ends of pq
z, w ------ rq
u, v ------ rp

In the projective plane the lines xz and yw meet in a, uz and wv meet in b.

We use the perspectivities centered at a and b to project pq isometrically onto rq, with q fixed and p going to p' on the ray $\overrightarrow{qr}$; and to project pr onto rq so that p goes onto p'' with r fixed and between p'' and q.

(over)

The proof is completed by showing that p' is either equal to p'' (this happens if and only if $a = b$ and the triples z, x, u and v, y, w are collinear) or between p'' and q. By a preliminary choice of the line at infinity it is possible to reduce to the case pictured, where a and b are not at infinity and are on the same side of pq as r is. The argument is straightforward but messy use of plane-ordering (a line splits a plane into two halfs with some obvious properties: cf. Borsuk & Smielew, Foundations of Geometry).

As a consequence of the condition for triangle equality, the triangle inequality is strict for all noncollinear triangles if and only if the boundary of the convex body does not have two coplanar, noncollinear liner segments.

Is there any function $F(x)$ such that

$$\int_0^x \frac{dt}{\sqrt{1+t^3}} = F(x) + \text{constant.}$$

yes

~~Meaningless as it stands~~

Of course, Look in any book on elliptic integrals. Say "Greenhill", for ex.

10-9-78

Clyde A Bridger

Spfld Ill

Leo Moser

Any set of (open) squares of total area 1 can be placed in a square of area 2 without overlap.
(i) Is the same true of any set of rectangles of total area 1 and longest edge 1 ?
(ii) What is the corresponding sharp result in higher dimensional space

Can the rectangles of side $\frac{1}{n}$, $\frac{1}{n+1}$, $n=1,2,3\ldots$ all be placed without overlap in a single unit square?

Do there exist integers $a_1<a_2<a_3$ and $b_1<b_2<\ldots<b_n$ (for every n) such that a_i+b_j is a perfect square for $i=1,2,3$ and $j=1,2,\ldots,n$?

D. Kleitman (Calg. Int. Conf.; Combinatorial Structures & Their Applications, 1970) showed $c_1\frac{2^{2n}}{n^{5/2}} \leqq$ this # $\leqq c_2\frac{2^{2n}}{n^{5/2}}$ (pp. 209-213).

A round-robin tournament on n players involves $\binom{n}{2}$ games each of which ends in a win for one of the players. Obtain asymptotic estimates for the number of distinct sets of ordered vectors $(S_1 \le S_2 \le \ldots \le S_n)$ which can arise as scores in such a tournament.

Reid & Parker (J. Comb. Theo. 9, 225-238, 1970) showed (contrary to a conj. of Erdős, Moser) $g(n) \geqq [\log_2(\frac{16n}{7})]$ for $n \geqq 14$.

Let $g(n)$ be the largest number such that every $n.n.$ tournament on n players contains a transitive ~~tour~~ subtournament on $g(n)$ players. It is conjectured that $g(n) \sim \log_2 n$.

Leo Moser

Is it true that for all n suff. large there exist n consecutive integers each of which has two prime factors in common with some other members of the set.

Can one place for any $n > 2$ the integers $1, 2, \ldots, \binom{n}{2}$ on the $\binom{n}{2}$ intersections of n lines in the proj. plane in such a way that the sum of numbers on each line is the same ?

Yes. It can be done with $n^{\log_2 7} \approx n^{2.71}$ See V. Strassen, ~~Nu~~ Numerische Mathematik 1969 (possibly 1970) E. Reingold (Dept of Computer Sci) Sept 2, 1970

Can one in general get by with fewer than n^3 multiplications in computing the product of 2 $n \times n$ matrices ?

Estimate the maximal number $f(n)$ of points which can be placed in a unit cube ~~sut~~ so that the distance between any pair is at least 1. $f(1) = 2$, $f(2) = 4$, $f(3) = 8$ and $f(4) = 17$. What is $f(5)$?

If ϵ_i $i = 1, 2, \ldots n$ is a sequence of ± 1's and $T(k) = \left| \sum_{i=1}^{n-k} \epsilon_i \epsilon_{i+k} \right|$ Does

$\min_{\epsilon_i} \max_{0 < k \leq n} T(k) \to \infty$ with n ?

Leo Moser

Do infinitely many powers of 2 miss the digit 1 in their decimal representation?

Are there an infinity of n such that in base 3 their digits are 0's and 1's and in base 5 their digits are 0's, 1's and 2's ?

Is $\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{(p-1)!} \equiv 0 \pmod{p}$ for more than 2 values of (prime) p ?

Prove or disprove - there exists a fixed constant c such that if a_{ij} are integers with $\max_{i,j} |a_{ij}| = \alpha$ and $n > m$ then the system of eqns $\sum_{j=1}^{n} a_{ij} x_j = 0$, $i = 1, 2, \dots m$ has a non-trivial solution with $|x_i| < (c\alpha)^{\frac{m}{n-m}}$.

Find or estimate the maximal number of (0,1) $n \times n$ matrices which commute in pairs.

Find or estimate the maximal number of (0,1) $n \times n$ matrices having exactly one ~~two~~ 1 per row which commute in pairs.

Find or estimate the maximal number of ordinary graphs on n vertices whose incidence matrices (vertices vs vertices) commute in pairs.

J. C. Ruppman

If $2 \cdot 2 \cdot 2 \cdots 2 \cdots = 2^N$

and $1 \cdot 2 \cdot 3 \cdot 4 \cdots N = N!$

and $2 \cdot 4 \cdot 6 \cdots 2N = 2^N N!$

what is

$1 \cdot 3 \cdot 5 \cdot 7 \cdots 2N+1 = ?$

$2^N = \prod_{i=1}^{N} 2_i \quad , \quad N! = \prod_{i=1}^{N} i$

$$\prod_{i=1}^{N} (2i+1) = (2N+1)! / 2^N N!$$

Take the sequence of ascending primes + construct a difference table of the absolute values of the differences, thus:

```
2
    1
3       1
    2       1
5       0       1
    2       2       1
7       2       2       1
    4       0       2       1
11      2       0       2
    2       0       0
13      2       0
    4       0
17      2
    2
19
```

Question: does this diagonal contain only ones after the first column?

S. Chilton

Note: a number of us tried for all the primes ≤ 1000. It works that far, at least barring any arithmetic mistakes we may have made.

The conjecture holds at least for all primes < 792,722 (see R.B. Killgrove and K.E. Ralston, "On a Conjecture Concerning the Primes," Math. Tables Aids Comput. 13 (1959), 121-122)

Karl K Norton

P. T. Bateman & P. Erdös

Does there exist a group of cardinal number $\mathfrak{m} > \aleph_0$ which has only $\mathfrak{m}$ subgroups?

Rotman: If G is abelian with $|G| = m > \aleph_0$, then G has 2^m subgroups.

~~The~~ One needs the existence of a subset X of G satisfying

(1) If $[X]$ is the subgroup generated by X, then every cyclic subgroup of G meets $[X]$;

(2) If $Y_1 \neq Y_2 \subset X$, then $[Y_1] \neq [Y_2]$.

When G is abelian, existence is provided by maximal independent subsets, where ~~set~~ X independent means $\sum m_i x_i = 0 \Rightarrow$ each $m_i x_i = 0$; $x_i \in X$, $m_i \in \mathbb{Z}$.

One could complete the nonabelian case similarly if he could find an X satisfying (1) and (2).

Nievergelt

If $s_1, s_2, \ldots$ is a sequence of real numbers such that

1) $|s_{n+1} - s_n| \leq \frac{1}{n}$ and
2) for all $k > 1$, the subsequence $s_k, s_{k^2}, s_{k^3}, \ldots$ converges

is it true that the sequence $s_1, s_2, \ldots$ also converges?

Can show that for all k, h,

$$\lim_{n\to\infty} s_{k^n} = \lim_{n\to\infty} s_{h^n}$$

5

Nievergelt

Let π be an equivalence relation of finite index on the set Z of positive integers such that for all $x, y \in Z$

$$x \,\pi\, y \Rightarrow (2x \,\pi\, 2y) \wedge (2x+1 \,\pi\, 2y+1) \wedge (3x \,\pi\, 3y) \wedge (3x+1 \,\pi\, 3y+1) \wedge (3x+2 \,\pi\, 3y+2)$$

Conjecture:

There exist N, k such that for all $x, y > N$,

$$x \,\pi\, y \text{ iff } x \equiv y \ (k)$$

5

T. McLAUGHLIN.

Note: The answer is no. Laver, Annals 7 Math., Jan. '71.

THE FOLLOWING IS A PROBLEM OF G.E. SACKS:

DOES THERE EXIST A (∞) COUNTABLE SET OF (∞) COUNTABLE LINEAR ORDERINGS WHICH IS INDEPENDENT, i.e., NO ONE OF THEM IS ORDER-IMBEDDABLE IN ANY FINITE ORDER SUM OF OTHERS? (THE DIFFICULTY IN THE PROBLEM IS DUE TO THE VAST HORDE OF POSSIBLE EMBEDDING FUNCTIONS)

T. McLaughlin

Independence Questions.

1. Can it be shown in ZF (i.e., Zermelo-Fraenkel set theory without the ax. of choice) that every compact metric space is separable? (I would guess the answer is no. The natural proof seems to rely quite critically on the countable ax. of choice.)

2. Can it be shown in ZF that an uncountable subset of a separable metric space has a condensation point? (Even in the simple case of Baire nullspace N^N, it seems necessary to assume the statement (which is independent of ZF) that countable unions of countable sets are countable.)

Jockusch

Let N be the set of all natural numbers, and let C, C_1, C_2 refer to collections of finite subsets of N.

<u>Defn</u> C is <u>dense</u> if ~~for~~ every infinite set $A \subset N$ contains some element of C as a subset.

<u>Defn</u> C is <u>extendible</u> if for every set $D \in C$ we have $(D \cup \{x\}) \in C$ for all but finitely many $x \in N$. (The finite number of exceptional x may depend on D.)

<u>Conjecture</u> If C_1, C_2 are each dense and extendible, then there is a set D which belongs to each of C_1, C_2.

(Note: If the conjecture is weakened by adding the following hypothesis, it becomes equivalent to Ramsey's theorem:

There is a (fixed) number n such that for every infinite set $A \subset N$, A contains as set D of cardinality at most n which belongs to $C_1 \cup C_2$.)

Note: (May 14, 1968) This conjecture has now been proved by Galvin and Prikry.

(F. I. Perticani)
Ago. 9. 1971

For $f(x,y)$ smooth, define

$$\tilde{f}(x,y) = \text{P.V.} \int_{-\infty}^{\infty} \frac{f(x-t, y-t^3)}{t}\, dt.$$

Is $f \longmapsto \tilde{f}$ continuous on $L^p(\mathbb{R}^2)$ for $p \neq 2$? (This means: there is a c_p such that $\|\tilde{f}\|_{L^p} \le c_p \|f\|_{L^p}$).

See the paper by Nagel, Rivière & Wainger in Amer. J. of Math (to appear).

J.E. Wetzel
3/72

What is the side of the equilateral triangle of least area that can accommodate each arc of length 1?

If this is too easy - what is the perimeter of the triangle with given angles and least perimeter that can accommodate each arc of length 1?

Of all such triangles, is the equilateral the smallest?

C. Bridger
2 - 73

~~Is~~ the set $x = \pm 8,\ y = \pm 3$ the only solution in integers for the (Diophantine) equation

$$5(y^2-1)^2 = x^2 + 256 \quad ?$$

C. Bridger
12 - 74

Edouard Lucas in his Theorie des Nombres p. 150 (1891 edition), writes

Verify the formula $\left(\frac{a-b}{c} + \frac{b-c}{a} + \frac{c-a}{b}\right)$

times $\left(\frac{c}{a-b} + \frac{a}{b-c} + \frac{b}{c-a}\right) = 9$

on supposing ~~that~~ $a+b+c=0$

What ~~is~~ the "trick" that will give a quick solution?

Solution outline
C. Bridger
2 - 76

The "trick" is to multiply out and set

$\left(\frac{a-b}{c}\right)\left(\frac{a}{b-c}\right) = x$, $\left(\frac{b-c}{a}\right)\left(\frac{b}{c-a}\right) = y$, $\frac{c-a}{b} \cdot \frac{c}{a-b} = z$

so that $xyz = 1$

Then $(x+1)(y+1)(z+1) + 1 = 9$ etc. to the end

Solution
March 2 1989

$xy = x$, $yz = y$, $z(x) = z$

$y = 1$ $z = 1$ $x = 1$

Assuming $x, y, z = 1$ / $x = y = z = 1$

$(1+1+1)(1+1+1)$
$(3)(3) = 9$

17 Oct. 1973.

Recently, Prof. Bateman established (Problem E2051, Amer. Math. Monthly 76 (1969), 190-19 that the arithmetic function $f_s(n) = (2s)^{-1} r_s(n$ is multiplicative iff $s \in \{1, 2, 4, 8\}$, where $r_s(n)$ is the number of representations of n as a sum of s integral squares. Prove the stronger result that $f_s(\cdot)$ is generalized multiplicative (for a definition see Amer. Math. Monthly 72 (1965), 1140-1141) if and only if $s \in \{1, 2, 4, 8\}$. Apply analogous methods to generalize the Bateman-Lagrange theorem; I.e., $\varphi_s(\cdot)$ is generalized multiplicative iff $s \in \{0, 1, 2, 3, \ldots, 8\}$ see Bateman, et al., Notices (Jan, 1973).

Consider the following synthesis of the Brouwer-Schauder fixed-point Th. & Minkowski Th. on the existence of lattice points. Determine necessary & sufficient conditions on a continuous transformation T of a compact set of E^n into itself in order that T has a fixed lattice point distinct from The Origin.

17 Oct. 73   Prove That there is ~~a~~ polynomial (over $\mathbb{Z}$) of degree at most 21 in 2 variables whose range is precisely the recursive set of all prime numbers.

(e.g.: Proc. AMS 4 (1953), 544-545.)

18 Oct 73.   Prof. L. Moser, among many others, showed that every real-valued monotone multiplicative arithmetic function $f(\cdot)$, $f(n) \not\equiv 0$, satisfies $f(n) = n^{\alpha}$ for some constant α. Prove the stronger result: every real-valued monotone generalized multiplicative arithmetic function $f(\cdot)$, $f(n) \not\equiv 0$, satisfies $f(n) = n^{\alpha}$ for some constant α. (See: Canadian Math. Bull. 9 (1966), pg. 115 Prob. 112.)

Same Ref. & Def.

mistake ← why?;

A. S. Fraenkel (~1974)

$\alpha_1, \dots, \alpha_m$ ~~rational~~ real, $r_1, \dots, r_m$ rational. Suppose that $[n\alpha_i + r_i]$ $\boxed{(1 \le i \le m)}$ $n=1,2,\dots$ are complementing, i.e., each positive integer appears exactly once in exactly one of the sequences $[n\alpha_i + r_i]$. (i) Prove or disprove: α_i/α_j = integer for some $i \ne j$ (if $m \ge 3$) the α_i & m may be equal says

$\frac{\alpha_i}{\alpha_j}$ is dependant upon the rational number assigned α_i & α_j in other words you may or may not have an integer

(ii) In this connection, consider the complementing system $\left[\frac{2^m-1}{2^{m-k}} n\right] + 1 - 2^{k-1}$, $k = 1, \dots, m$, $n = 1, 2, \dots$. Erdős asked whether this is ~~perhaps~~ the only compl. system for which all the moduli α_i are different ($m \ge 3$).

Addition of June, 1991. 1. ~~Special~~ Proofs of special cases of the conjectures ~~(note that (i) ⟹ (ii))~~ have appeared in Fraenkel J Comb Theory A 14 (1973) 8-20.

2. The validity of ~~(ii) for~~ (i) has been shown by Graham J. Comb. Theory A (15) ← (1973) 354-358. (Two moduli must be equal, as in the integer case.)

3. Special cases of both conjectures (rational moduli) were proved in Berger, Felzenbaum, Fraenkel, J Comb Theory A 42 (1986) 150-153.

4. Morikawa, Bull. Liberal Arts, Nagasaki Univ 26 (1985) 1-13, motivated by the conjectures, has given a proof of the "Japanese Remainder Theorem" giving a nec & suff condition for the disjointness of 2 rational Beatty sequences. (The special case when both denominators are 1 is the Chinese Remainder Thm) Simpson, Tech Report 3/90, Curtin Univ, Perth WA (1990) has given a more transparent proof. He and Vera Sós are considering to write up a proof based on the 3-gap thm.

5. Simpson, Discr. Math (≥1991) has proved Conj (i) for the special case when the smallest rational modulus is ≤2. *Terminology due to Simpson.

6. The general case of both conjectures is still open for the case of rational moduli, though it is settled for both the integer & irrational case. A. S. Fraenkel.

G. S. Sacerdote
July 75

① Is every countable metabelian group embeddable into a finitely presented group?

② If G is a countable group all of whose finitely generated subgroups are 3-manifold groups, is G embeddable into a finitely presented group?

...r the case where all
...le α_i are irrational

[illegible]
[illegible]

5

(A May) Prove

$$\lim_{n\to\infty} e^{-n\gamma}\left(1+\frac{n\gamma}{1!}+\frac{(n\gamma)^2}{2!}+\dots+\frac{(n\gamma)^{n-1}}{(n-1)!}\right)$$

$$= \begin{cases} 1 & 0 \le \gamma < 1 \\ \frac{1}{2} & \gamma = 1 \\ 0 & 1 < \gamma \end{cases}$$

2. Jerrard 1977 Conjecture: If

$$S^2 \xrightarrow{f} S^2 \xrightarrow{g} E^2$$

with f, g continuous, $\exists\, x$ such that $g(x) = g \cdot f(x)$

(1) ~~General~~ case: **CHALLENGE!**

(I need this solved)

$b_i(t)$, $d_i(t)$, $\mu_1(t)$, $\mu_2(t)$ nonnegative analytic functions of t for $t \geq 0$, $Q = Q(s_1, s_2, s_3, t)$.

Solve the following partial differential equation:

$$\frac{\partial Q}{\partial t} = \Big\{ s_1^2 b_1(t) + s_1(s_2 - 1)\mu_1(t) + s_1(s_3 - 1)\mu_2(t) - s_1\left[b_1(t) + d_1(t)\right] + d_1(t) \Big\} \frac{\partial}{\partial s_1} Q$$

$$+ \Big\{ s_2^2 b_2(t) - s_2\left[b_2(t) + d_2(t)\right] + d_2(t) \Big\} \frac{\partial}{\partial s_2}$$

$$+ \Big\{ s_1^2 b_3(t) - s_3\left[b_3(t) + d_3(t)\right] + d_3(t) \Big\} \frac{\partial}{\partial s_3} Q,$$

$Q(s_1, s_2, s_3, 0) = s_1^{m_{10}} s_2^{m_{20}}$, m_{10} and m_{20} are fixed constants.

If you can solve this, please call Emy Tan.
~~[illegible]~~
332-4428
Thanks!

(2) Special case:

If the above PDE is not solvable, try to solve the special case.

$b_i(t) = b_i$, $d_i(t) = d_i$, $\mu_1(t) = \mu_1$ and $\mu_2(t) = \mu$
all indep. of t.

12

McConnell 1978

Let $n_1, n_2, \ldots, n_k$ be relatively prime square free integers. Let $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_k$ be any sequence of ± 1 s. Is it true that there exists a prime p s.th.

$$\left(\frac{n_i}{p}\right) = \varepsilon_i \qquad i = 1, 2, \ldots k.$$

Theory of the Fabric of Time

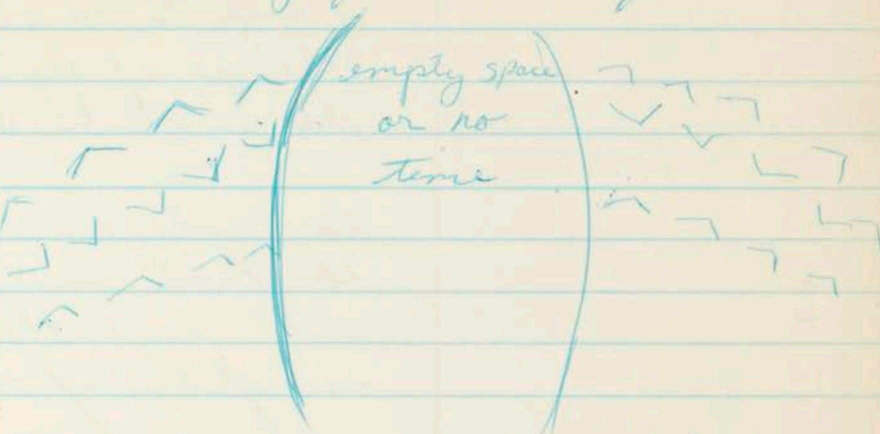


diagram 1

Mary Seibert Johnson
1984
October 8

arch 2, 1989
Is empty space or
no time constant
with a null set
i.e. nonexistant
or nullification
of previous time?
or time which
might have been

think of time as a fabric

in its orginal existance
it would consist of a sheet (similar to a plane)
which could not be distinguished
from the normal

as time rearranges the above
diagram occurs:

time splits apart
leaving a hole
which represents
empty space &/or no time

the fabric on each side
appears to break into rt L's
which reorient

the ∠ of the curvature
of time is consistent with
the pt of segment from
the parabolic hole

in realignment
the fabric interweaves or slips
under each other to form the
new fabric of time

similar to liquid crystals

Nobel Prize Nominee, 1983 Medicine
Winner 1984 Medicine
Nobel Prize Nominee, 1985, 86
Medicine, Chemistry
Nobel Prize Winner 1986 awarded
Chemistry Jan. 1987

Pulitzer Prize Winner 1987
awarded Jan or Feb 1988

March 2, 1989

parallel existances in time paths is a reality

Time through the parabolic hole can alter parallel
existances into the same one
or
Two parallel time paths may merge

5

starting with the affirmation of man I
work my way backwards using cynicism
the time monitor - the space measurer
live sweat - dream light years
the tidor the rise and the fall

11-6-89'

1. Bromley HALL APPLIED
2.
3.
4.

N Boston
August, 1990

Proved by
Tuval Foguel.
See J. Alg. 1991

(1) Let G be a group and $A = \mathrm{Aut}(G)$. Given subgroup H of G let $H^* = \{\alpha \in A : h^\alpha = h \ \forall h \in H\}$. Given subgroup B of A let $B^* = \{g \in G : g^\beta = g \ \forall \beta \in B\}$. Call G Galois-theoretic (GT, for short) if both $*$ maps are bijective.

Q^n Are the only GT groups $\{e\}, C_3, S_3, \cancel{C_3 \times S_3}$.

Note: Marty Isaacs has shown that a finite GT group must be solvable.

(2) Let $d(G)$ denote the number of generators of a finite group G. A q^n important in profinite group theory (due to Ribes et al.) is whether, given 2 finite groups G, H, there exists a finite group K containing copies of G and H and generated by these copies, such that $d(K) = d(G) + d(H)$.

(3) Baker et al. have asked whether there is a 5th term in the sequence 1, 3, 8, 120, i.e. an integer which when multiplied by 1, 3, 8, or 120 gives a square minus one. (This problem has a lot of algebraic structure to it, maybe surprisingly.)

Sorry, this is known. See e.g. Fibonacci Quart. 17.

3-10-97
Clark Kimberling
CK6@evansville.edu

Use any sequence $c_1, c_2, c_3, \ldots$ of 0's and 1's to define a <u>repetition-resistant</u> <u>sequence</u> $s = (s_1, s_2, s_3, \ldots)$ inductively as follows:

(i) $s_1 = c_1$, $s_2 = 1 - s_1$;

(ii) for $n \geq 2$, let

$$L = \max\{ i \geq 1 : (s_{m-i+2}, \ldots, s_m, s_{m+1}) = (s_{n-i+2}, \ldots, s_n, 0) \text{ for some } m < n \},$$

$$L' = \max\{ i \geq 1 : (s_{m-i+2}, \ldots, s_m, s_{m+1}) = (s_{n-i+2}, \ldots, s_n, 1) \text{ for some } m < n \}$$

(so that L is the maximal length of tail-sequence of $(s_1, s_2, \ldots, s_n, 0)$ that already occurs in $(s_1, s_2, \ldots, s_n)$, and similarly for L'), and

$$s_{n+1} = \begin{cases} 0 & \text{if } L < L' \\ 1 & \text{if } L > L' \\ c_n & \text{if } L = L'. \end{cases}$$

<u>Prove</u> or <u>disprove</u>: <u>s contains every binary word.</u>

Example: If $c_i = 0$ for all i, then

$S = (0,1,0,0,0,1,1,0,1,0,1,1,1,0,0,1,0,0,1,1,1,1,0,1,1,0,\ldots)$