4-color-seminar / 1 Definitions (abbreviated vocabulary)

(Haken on Shimamoto's construction; October 20, 1971)

Special conventions:

Graph: finite, planar, non-oriented graph without loop-edges.

Triangulation: Graph as above which can be embedded into the 2-sphere in such a way that each connected component of 2-sphere minus graph is a triangle.

4-coloration: Assignment of one of four colors, named 1,2,3,4, to each vertex of a graph so that any two vertices which are joined by an edge (of the graph) obtain different colors.

Critical graph: Graph as above which i) does not admit a 4-coloration ii) has the property that if one arbitrary edge is subtracted then a 4-colorable graph is obtained.

Minimal triangulation: Triangulation which is not 4-colorable such that every triangulation with fewer vertices is 4-colorable.

Configuration: Graph triangulating a disk (so that the boundary of the disk consists of edges and vertices of the graph), but not being a single triangle.

Boundary circuit of a configuration : Graph consisting of the boundary edges and vertices of that configuration.

Complementary configuration proper (of a configuration H in a triangulation T where H is a sub-graph of T): The configuration different from H which is a sub-graph of T and has the same boundary circuit as H.

Equivalence of two 4-colorations C_1, C_2 on same graph G: C_2 is obtained from C_1 by a permutation of colors (1,2,3,4).

Note: We do not consider equivalent colorations as "equal".

αβ-Kempe chain (in a 4-colored graph G, where α,β are any two different colors 1,2,3,or 4): A connected component of the sub-graph $G_{\alpha\beta}$ of G which consists of all those vertices of G which have color α or color β and of all edges of G which join two such vertices.

Note: A single vertex may be a Kempe chain.

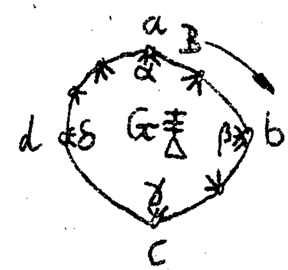
Degree of a vertex in a graph G: number of edges of G originating at that vertex.

First neighborhood of a graph G in a triangulation T (G a subgraph of T): The sub-graph N of T which contains precisely those vertices of T which belong to G or are edge-connected to vertices of G, and all edges of T which join two vertices of N.

4-color-seminar / 2   Kempe chain-argument and D-Reduction

1st Kempe chain theorem: Let G be a graph with a 4-coloration C. Let $K_{\alpha\beta}$ be an $\alpha\beta$-Kempe chain in (G,C) (=graph G 4-colored by C). Then exchanging the colors α and β on $K_{\alpha\beta}$ (and leaving the colors on all other vertices of G fixed) yields a ~~different~~ 4-coloration $C' \neq C$ of G.

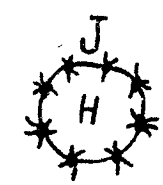
2nd Kempe chain theorem: Let G be a configuration with boundary circuit B and with a 4-coloration C. Let a,b,c,d be four distinct vertices lying in that order on B (see Fig.) and having the four different colors $\alpha, \beta, \gamma, \delta$. Let $K_{\alpha\gamma}$ be an $\alpha\gamma$-Kempe chain ~~whin~~ (G,C) which contains both vertices a and c. Then there does not exist a $\beta\delta$-Kempe chain in (G,C) which contains both vertices b and d.



Proof: 1st theorem immediate from definition of 4-coloration and Kempe chain. 2nd theorem immediately from planarity of G and connectedness of Kempe chains.

Kempe chain-argument means: deriving new colorations C_1, C_2, ... of graph G from a given coloration C by iterated use of the above theorems. In particular, if G is a configuration with boundary circuit B, deriving colorations which induce different colorations on B.

Abbriviated definition of D-reducibility:



A configuration H is called D-reducible if the following holds.

Assume that H is sub-graph of a triangulation T and denote by J the complementary configuration of H in T. Further assume that there exists a 4-coloration C of J . Then these assumptions allow to conclude by Kempe chain-argument applied to configuration J (and by nothing else) that there exists a 4-coloration C^* of J which can be extended to a 4-coloration of T.

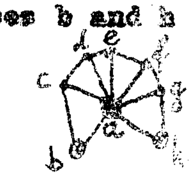
For more details see Heesch: Untersuchungen zum Vierfarbenproblem, Kapitel I.

For a simple example of D-reduction see page 5 of these notes.

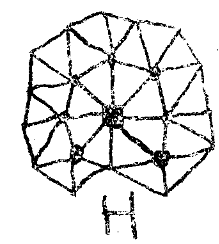
4-color-seminar / 3 Statement of the main theorem

(Haken on Shimamoto's construction; October 20, 1971)

Theorem A: If the 4-color-conjecture is false then there exists a critical triangulation A which contains an 85666665-horseshoe; by this we mean a sub-graph which consists of a vertex a of degree 8 and seven neighbor vertices b, c, d, e, f, g, h of a (lying in that order around a) where vertices b and h have degree 5 and vertices c, d, e, f, g have degree 6 (see Fig.).



Theorem B: The configuration H which is a first neighborhood of a 85666665-horseshoe in a triangulation (see Fig.) is D-reducible. (The boundary circuit of H has 14 vertices.)



Theorem C: A critical triangulation cannot contain a D-reducible configuration.

Theorems A, B, C imply that the 4-color-conjecture is true.

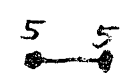
The proof of Theorem C is an immediate consequence of the definitions: Assume D-reducible configuration H lies in critical triangulation T. Then complementary configuration J of H in T contains less edges than T and thus (by def. of criticality) possesses a 4-coloration C. Then (by def. of D-reducibility) T itself possesses a 4-coloration (which extends a 4-coloration C^* of J that was derived from C by Kempe chain-argument). This is a contradiction (by def. of criticality). Q.E.D.

The proof of Theorem B is given by machine-computation. The computation was done on April 24, 1968 at BNL and will be checked by different programs on different machines.

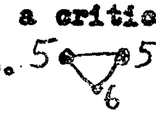
The proof of Theorem A is given in this seminar following Shimamoto's construction: Graph A is constructed in the following steps.

Theorem A_0: If the 4-color-conjecture is false then there exists a critical triangulation A_0 which contains a vertex of degree 5.

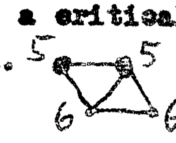
Theorem A_1: If the 4-color-conjecture is false then there exists a critical triangulation A_1 (derived from A_0) which contains a 55-edge (i.e., an edge joining two vertices of degree 5).



Theorem A_2: If the 4-color-conjecture is false then there exists a critical triangulation A_2 (derived from A_1) which contains a 556-triangle.



Theorem A_3: If the 4-color-conjecture is false then there exists a critical triangulation A_3 (derived from A_2) which contains a 5566-diamond.



Finally Graph A is derived from A_3.

4-color-seminar / 4 Ancient theorems proving Theorem A_0 (Part 1)

(Haken on Shimamoto's construction, October 20, 1971)

Theorem 1: If the 4-color-conjecture is false then we have the following.

(i) There exists a (planar) graph G which is not 4-colorable.

(ii) There exists a (planar) triangulation T which is not 4-colorable.

(iii) There exists a minimal triangulation T_{min} (as defined on p.1).

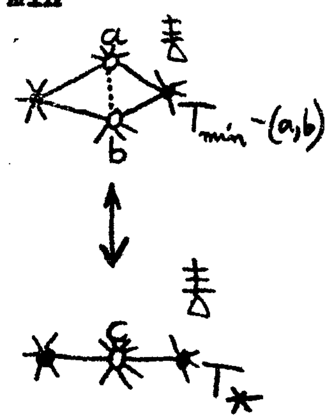
(iv) There exists a critical triangulation (as defined on p.1). In fact, every minimal triangulation is critical, but not vice versa.

Proof: (i) follows from the fact that every map (in the sense of geography) has a dual graph G (the vertices of G may be regarded as the capitals of the countries of the map, the edges may be regarded as direct roads joining the capitals of two countries which have a common border). A proper 4-coloration of a map (as considered in the 4-color-conjecture) induces a 4-coloration of the dual graph G and vice versa.

(ii) follows from the fact that graph G of (i) can be completed to become a triangul, by adding edges ~~to a triangulation T as demanded.~~ (G is embedded into the 2-sphere, and if any connected component of 2-sphere minus G is not a triangle then it is triangulated by additional diagonal edges.)

(iii) follows immediately from the fact that the triangulation T of (ii) has only finitely many vertices.

In order to prove (iv) we have to show: If (a,b) is an arbitrary edge in T_{min} of (iii) (with endpoints denoted by a and b) then T_{min} - (a,b) is 4-colorable. We do this by observing that T_{min} - (a,b) is a configuration with boundary circuit B containing four boundary vertices (see Fig.). We obtain a triangulation T_* from T_{min} - (a,b) by "contracting" B (see Fig.) to a pair of edges (with common vertex c obtained from a and b by identification). Now T_* has one vertex less than T_{min} and thus (by definition of minimality) posesses a 4-coloration C. Now reversing the contraction ("cutting T_* along the pair of edges" and "splitting" vertex c into vertices a, b) we obtain a 4-coloration C' of T_{min} - (a,b) (where a and b have the same color which vertex c had according to C). Q.E.D.



Theorem 2: If T is an arbitrary triangulation (of the 2-sphere) then

(2.i) T does not contain any vertex of degree 0 or 1.

(2.ii) T contains at least one vertex of degree <6.

4-color-seminar / 5 Ancient theorems proving Theorem A_Ω (Part 2)

(Haken on Shimamoto's construction; October 20, 1971)

Proof of Theorem 2: (2.i) follows immediately from the definition of triangulation.

(2.ii) is a consequence of Euler's formula

$$v + t - e = 2,$$

where v, t, e are the number of vertices, triangles, edges in T. Since each edge is border of 2 triangles and each triangle has 3 border edges wem have $t = \frac{2}{3} e$ and hence

$$v - \frac{1}{3} e = 2.$$

Let $v_2, v_3, \ldots, v_m$ be the number of vertices of degree $2, 3, \ldots, m$ in T where m is the greatest degree that actually occurs in T. Then we have

$$v = v_2 + v_3 + \ldots + v_m \quad \text{and}$$

$$e = \frac{1}{2}(2v_2 + 3v_3 + \ldots + mv_m).$$

Substituting these values for v and e we obtain

$$v_2 + v_3 + \ldots + v_m - \frac{1}{6}(2v_2 + 3v_3 + \ldots + mv_m) = 2, \quad \text{or}$$

$$4v_2 + 3v_3 + 2v_4 + v_5 - v_7 - 2v_8 - 3v_9 - \ldots - (m-6)v_m = 12.$$

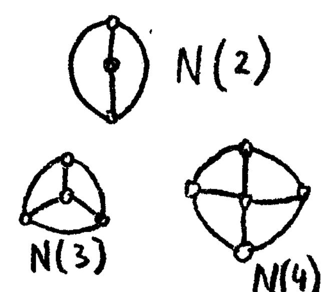
Hence, not all of v_2, v_3, v_4, v_5 can be zero. Q.E.D.

Theorem 3: The following configurations are D-reducible.

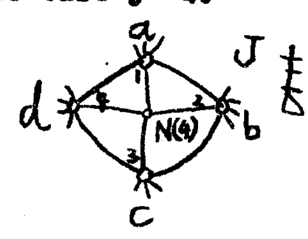
(3.i) A first neighborhood N(2) of a vertex of degree 2 (in a triangulation).

(3.ii) A first neighborhood N(3) of a vertex of degree 3.

(3.iii) A first neighborhood N(4) of a vertex of degree 4.



Proof: Assume N(j) lies in a triangulation T and its complementary configuration, J, in T possesses a 4-coloring C. The boundary circuit B of N(j) has j vertices. To prove the theorem we have to derive a coloration C^* of J which can be extended over N(j). In the case that $j < 4$ we simply take $C^* = C$. This induces a coloration on B in which at most 3 colors occur; thus we can extend this coloration over N(j) by assigning the 4^{th} color to the interior vertex of N(j). So we are left with the case $j = 4$. Denote the 4 boundary vertices of N(4) by a, b, c, d (see Fig.). In the case that C induces a coloration on B in which only 2 or 3 colors occur then we take

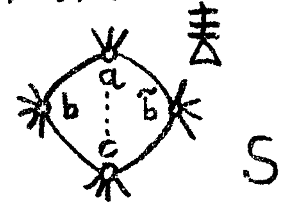


(Haken on Shiramoto's construction; October 20, 1971)

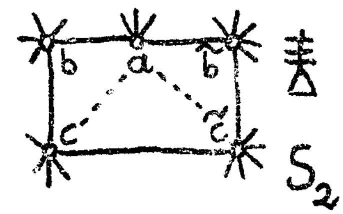
again $C^* = C$. In the case that C induces a coloration on B in which all 4 colors occur ~~then~~ we use the Kempe chain-argument as follows. There is a coloration C' of J which is equivalent to C and which assigns color 1 to a, color 2 to b, color 3 to c, color 4 to d. Now we distinguish two cases: Case 1: There is no 13-Kempe chain that contains both vertices a and c in (J,C'). In this case we obtain C^* from C' by exchanging the colors 1,3 in the 13-Kempe chain (in (J,C')) which contains vertex a. Case 2: There is a 13-Kempe chain in (J,C') that contains both vertices a and c. In this case we use the 2nd Kempe chain theorem for concluding that there is no 24-Kempe chain that contains both vertices b and d. Then we derive C^* from C' by exchanging the colors 2,4 in the 24-Kempe chain that contains vertex b. In each of the two cases C^* induces on B a coloration which uses only 3 colors. Thus C^* is extendable over N(4). This finishes the proof of Theorem 3.

Proof of Theorem A_o: We claim that the critical triangulation of Theorem 1.(iv) contains a vertex of degree 5 and thus can be taken for A_o. By Theorem 2.(i),(ii) the triangulation contains at least one vertex of degree 2,3,4, or 5. By Theorem 3 and Theorem C the critical triangulation cannot contain any vertex of degree 2,3, or 4. This proves Theorem A_o.

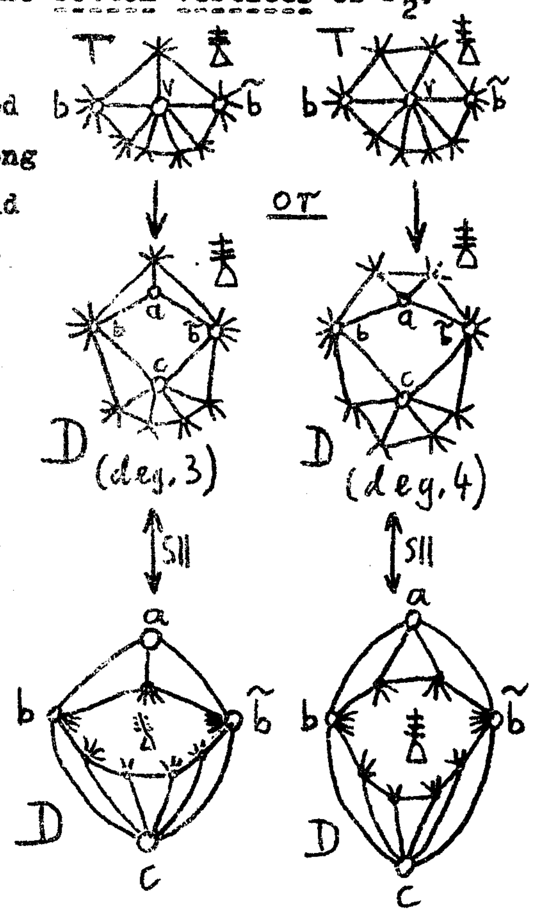
(Haken on Shimamoto's construction; October 21, 1971)



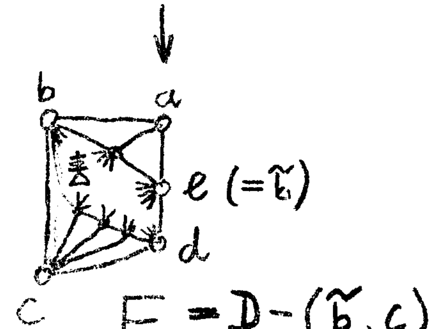
An S-graph is a configuration S which is obtained from a critical triangulation T by deleting one edge, say (a,c). The vertices a and c on the boundary circuit of S are then called the special vertices of S. (See Fig.)



An S_2-graph is a configuration S_2 which is obtained from a critical triangulation T by deleting two edges bordering to one triangle. We denote the vertices so that the deleted edges are (a,c) and (a,c̃) (see Fig.). Then vertex a is called the top vertex of S_2, and vertices c and c̃ are called the bottom vertices of S_2.



A D-graph is a configuration D which is obtained from a critical triangulation T by "cutting along two consecutive edges", say (b,v) and (v,b̃), and thus "splitting the vertex v" into two vertices a and c in such a way that vertex a is either of degree 3 or of degree 4 in D (see Fig.). (In all applications we shall depict a D-graph as in the lower part of the Fig. which is obtained from the upper part by an involution about the boundary circuit of D.) Vertex a is called the top vertex of D, and vertex c is called the bottom vertex of D.



An E-graph is a configuration which is obtained from a D-graph D by deleting one edge, say (c,b̃) from the boundary circuit of D which originates from the bottom vertex, c, of D. (See Fig.; for practical purposes we have renamed vertex b̃ into e where it occurs in the E-graph E;) Again, we call vertices a and c the top and bottom vertex of E-graph E.

(Haken on Shimamoto's construction; October 21, 1971)

Coloration theorem for S-graphs: Let S be an S-graph with special vertices a and c. Denote the vertices in the boundary circuit of S by $\begin{smallmatrix} & a & \\ b & & d \\ & c & \end{smallmatrix}$ (Fig.p.7). Then we have the following:

(S.1) S is 4-colorable.

(S.2) If C is a 4-coloration of S then the special vertices a and c have the same color.

(S.3) If C is a 4-coloration of S such that a and c are colored 1, then in (S,C) there are a 12-Kempe chain, a 13-Kempe chain, and a 14-Kempe chain each of which contains both vertices a and c.

(S.4) S admits a coloration inducing $\begin{smallmatrix} & 1 & \\ 2 & & 2 \\ & 1 & \end{smallmatrix}$ (on the boundary circuit of S).

(S.5) S admits a coloration inducing $\begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix}$.

(S.6) S - (interior edge) admits (Case 1) a color. ind. $\begin{smallmatrix} & 1 & \\ 3 & & 3 \\ & 2 & \end{smallmatrix}$ or (Case 2) a color. ind. $\begin{smallmatrix} & 1 & \\ 3 & & 4 \\ & 2 & \end{smallmatrix}$.

(S.7) S - (a,b) admits a coloration inducing $\begin{smallmatrix} & 1 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix}$.

(S.8) S - (b,c) admits a coloration inducing $\begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 1 & \end{smallmatrix}$.

Coloration theorem for D-graphs:
Let D be a D-graph with boundary circuit $\begin{smallmatrix} & a & \\ b & & d \\ & c & \end{smallmatrix}$ (Fig.p.7; a = top vertex).
Then we have the following:

(D.1) D is 4-colorable.

(D.2) If C is a 4-coloration of D then vertices a and c have different colors.

(D.3) If C is a 4-coloration of D such that a, c are colored 1, 2 then in (D,C) there is a 12-Kempe chain which contains both vertices a and c.

(D.4) D admits a coloration inducing $\begin{smallmatrix} & 1 & \\ 3 & & 3 \\ & 2 & \end{smallmatrix}$.

(D.5) D admits a coloration inducing $\begin{smallmatrix} & 1 & \\ 3 & & 4 \\ & 2 & \end{smallmatrix}$.

(D.6) D - (interior edge) admits (Case 1) a color. ind. $\begin{smallmatrix} & 1 & \\ 2 & & 2 \\ & 1 & \end{smallmatrix}$ or (Case 2) a color. ind. $\begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix}$.

(D.7) D - (a,b) admits a coloration ind. $\begin{smallmatrix} & 1 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix}$.

(D.8) D - (b,c) admits a coloration inducing $\begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 2 & \end{smallmatrix}$.

(Haken on Shimamoto's construction; October 21, 1971)

Coloration theorem for S_2-graphs:

Let S_2 be an S_2-graph with boundary circuit $\begin{smallmatrix} b & a & \tilde{b} \\ c & & \tilde{c} \end{smallmatrix}$ (Fig.p.7; a = top).

Then we have the following:

(S_2.1) S_2 is 4-colorable.

(S_2.2) If C is a 4-coloration of S_2 then one of the bottom vertices c, $\tilde{c}$ has the same color as the top vertex a.

(S_2.3) If C is a 4-coloration of S_2 such that a and c are colored 1, and $\tilde{c}$ is colored 2 then in (S_2, C) there are a 13-Kempe chain and a 14-Kempe chain each of which contains both vertices a and c.

(S_2.4) S_2 admits a coloration inducing $\begin{smallmatrix} 2 & 1 & 3 \\ 1 & & 2 \end{smallmatrix}$.

(S_2.5) S_2 admits a colorat. ind. $\begin{smallmatrix} 3 & 1 & 3 \\ 1 & & 2 \end{smallmatrix}$.

(S_2.6) S_2 admits a coloration inducing $\begin{smallmatrix} 4 & 1 & 3 \\ 1 & & 2 \end{smallmatrix}$.

(S_2.7) S_2 - (interior edge) admits (Case 1) a coloration inducing $\begin{smallmatrix} 4 & 1 & 4 \\ 2 & & 3 \end{smallmatrix}$ or (Case 2) a color. ind. $\begin{smallmatrix} 3 & 1 & 2 \\ 2 & & 3 \end{smallmatrix}$ or (Case 3) a col. ind. $\begin{smallmatrix} 3 & 1 & 4 \\ 2 & & 3 \end{smallmatrix}$ or (Case 4) a col. ind. $\begin{smallmatrix} 4 & 1 & 2 \\ 2 & & 3 \end{smallmatrix}$.

(S_2.8) S_2 - (a,b) admits (Case 1) a col. ind. $\begin{smallmatrix} 1 & 1 & 2 \\ 2 & & 3 \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 1 & 1 & 4 \\ 2 & & 3 \end{smallmatrix}$.

(S_2.9) S_2 - (b,c) admits (Case 1) a col. ind. $\begin{smallmatrix} 2 & 1 & 2 \\ 2 & & 3 \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 2 & 1 & 4 \\ 2 & & 3 \end{smallmatrix}$.

(S_2.10) S_2 - (c,$\tilde{c}$) admits (Case 1 a col. ind. $\begin{smallmatrix} 3 & 1 & 3 \\ 2 & & 2 \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 3 & 1 & 4 \\ 2 & & 2 \end{smallmatrix}$.

(Haken on Shimamoto's construction; October 21, 1971)

Coloration theorem for E-graphs:

Let E be an E-graph with boundary circuit $\begin{smallmatrix} b & a \\ & e \\ & d \\ c & \end{smallmatrix}$ (Fig.p.7; a = top vertex, c = bottom vertex).

Then we have the following:

(E.1) E is 4-colorable.

(E.2) If C is a 4-coloration of E then a and c have different colors.

(E.3) If C is a 4-coloration of E such that a,c are colored 1,2 then in (E,C) there is a 12-Kempe chain which contains both a and c.

(E.4) E admits a coloration inducing $\begin{smallmatrix} 3 & 1 \\ & 3 \\ & 1 \\ 2 & \end{smallmatrix}$.

(E.5) E admits a col. ind. $\begin{smallmatrix} 3 & 1 \\ & 3 \\ & 4 \\ 2 & \end{smallmatrix}$.

(E.6) E admits a coloration inducing $\begin{smallmatrix} 3 & 1 \\ & 4 \\ & 1 \\ 2 & \end{smallmatrix}$.

(E.7) E admits a col. ind. $\begin{smallmatrix} 3 & 1 \\ & 4 \\ & 3 \\ 2 & \end{smallmatrix}$.

(E.8) E − (interior edge) admits (Case 1) a col. ind. $\begin{smallmatrix} 2 & 1 \\ & 2 \\ & 3 \\ 1 & \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 3 & 1 \\ & 2 \\ & 3 \\ 1 & \end{smallmatrix}$ or (Case 3) a col. ind. $\begin{smallmatrix} 4 & 1 \\ & 2 \\ & 3 \\ 1 & \end{smallmatrix}$.

(E.9) E − (b,e) admits (Case 1) a col. ind. $\begin{smallmatrix} 2 & 1 \\ & 2 \\ & 4 \\ 2 & \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 2 & 1 \\ & 3 \\ & 1 \\ 2 & \end{smallmatrix}$.

(E.10) E − (a,b) admits (Case 1) a col. ind. $\begin{smallmatrix} 1 & 1 \\ & 2 \\ & 4 \\ 2 & \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 1 & 1 \\ & 3 \\ & 1 \\ 2 & \end{smallmatrix}$.

(E.11) E − (c,d) admits (Case 1) a col. ind. $\begin{smallmatrix} 2 & 1 \\ & 2 \\ & 1 \\ 1 & \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 3 & 1 \\ & 2 \\ & 1 \\ 1 & \end{smallmatrix}$.

(E.12) E − (d,e) admits (Case 1) a col. ind. $\begin{smallmatrix} 2 & 1 \\ & 2 \\ & 2 \\ 1 & \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 3 & 1 \\ & 2 \\ & 2 \\ 1 & \end{smallmatrix}$.

(E.13) E − (d,e) admits (Case 1) a coloration ind. $\begin{smallmatrix} 3 & 1 \\ & 3 \\ & 3 \\ 2 & \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 4 & 1 \\ & 3 \\ & 3 \\ 2 & \end{smallmatrix}$.

(E.14) E − (e,a) admits (Case 1) a col. ind. $\begin{smallmatrix} 2 & 1 \\ & 1 \\ & 2 \\ 1 & \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 3 & 1 \\ & 1 \\ & 2 \\ 1 & \end{smallmatrix}$.

(E.15) E − (e,a) admits (Case 1) a coloration ind. $\begin{smallmatrix} 3 & 1 \\ & 1 \\ & 3 \\ 2 & \end{smallmatrix}$ or (Case 2) a col. ind. $\begin{smallmatrix} 4 & 1 \\ & 1 \\ & 3 \\ 2 & \end{smallmatrix}$.

 Proof of Coloration theorem S

(Haken on Shimamoto's construction; October 22, 1971)

Proof of (S.1): Since S = critical graph minus edge it is 4-colorable.

Proof of (S.2): Assuming the contrary, T = S + (a, c) would be 4-colorable, and thus not critical. Contradiction.

Proof of (S.3): Assuming the contrary, for at least one value of $\alpha = 2, 3, 4$ the 1α-Kempe chain in (S, C) which contains vertex c would not contain vertex a. Then exchanging colors $1, \alpha$ in that chain would yield a 4-coloration contradicting (S.2).

Proof of (S.4): By (S.1) and (S.2) there is a 4-coloration of S which colors a and c the same. Thus there is an equivalent 4-coloration C of S which induces either $\begin{smallmatrix} & 1 & \\ 2 & & 2 \\ & 1 & \end{smallmatrix}$ or $\begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix}$. In the first case (S.4) is satisfied; in the second case, by (S.3) there is a 14-chain from a to c, and thus (By the 2nd Kempe chain theorem) the 23-chain (in (S, C)) which contains $\breve{b}$ does not contain b; then exchanging colors 2, 3 in that 23chain yields a coloration of S as demanded in (S.4). This proves (S.4).

Proof of (S.5): By (S.4) there is a coloration C of S inducing $\begin{smallmatrix} & 1 & \\ 2 & & 2 \\ & 1 & \end{smallmatrix}$. Because of (S.3) the 23-chain containing $\breve{b}$ does not contain b; thus exchanging 2 and 3 in that chain yields the demanded coloration.

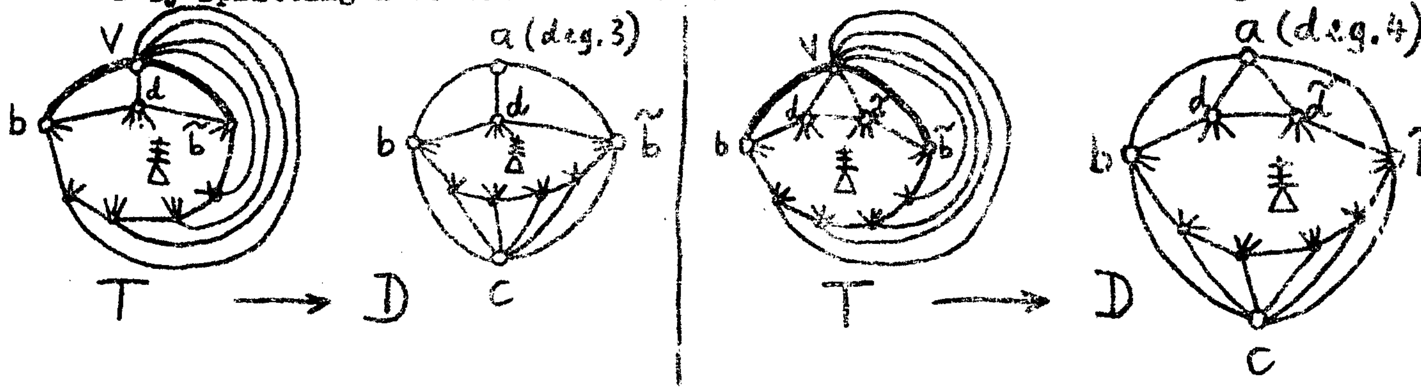
Proof of (S.6): Let (x, y) be an arbitrary interior edge of S, (i.e., an edge of S that does not belong to the boundary circuit of S, but may have an end point on that boundary circuit). Recall that S + (a, c) is a critical triangulation T. Thus T - (x, y) admits a 4-coloration, say C. This induces a coloration, for simplicity also called C, on T - (x, y) - (a, c) = S - (x, y) which gives different colors to a and c. Then C is equivalent to a coloration of S - (x, y) which induces one of the colorations $\begin{smallmatrix} & 1 & \\ 3 & & 3 \\ & 2 & \end{smallmatrix}$, $\begin{smallmatrix} & 1 & \\ 3 & & 4 \\ & 2 & \end{smallmatrix}$. Q.E.D.

Proof of (S.7): T = S + (a, c) is critical. Thus T - (a, b) admits a 4-coloration C; then C colors a and b with the same color (since otherwise T itself were 4-colorable). We regard C also as a coloration of S - (a, b) = T - (a, b) - (a, c). Now C is equivalent to a coloration of S - (a, b) as demanded.

Proof of (S.8): Same as proof of (S.7) with roles of a and c exchanged.

(Haken on Shimamoto's construction; October 22, 1971)

Proof of (D.1): Recall that D is obtained from a critical triangulation T by splitting a vertex v of T into two vertices a and c (see Fig. below).



Case 1: a is of degree 3 in D. Denote the interior vertex of D which neighbors a by d; (same notation in T). By deleting vertex a (and the 3 edges originating from it) from D yields a graph D - a which is "same" as T - (v,d) (i.e., the vertices and edges of D - a are in 1-1 correspondence with the vertices and edges of T - (v,d), c corresponding to v). Since T is critical T - (v,d) admits a coloration; denote the corresponding coloration of D - a by C. Now C can be extended to a coloration C' of D (since vertex a can be given a color different from the colors of b, d, $\tilde{b}$) Q.E.D.

Case 2: a is of degree 4 in D. Denote the interior vertices of D which neighbor a by d and $\tilde{d}$ (neighboring b and $\tilde{b}$, respectively). Now graph D - a is same as T - (v,d) - (v,$\tilde{d}$). Thus D - a admits a 4-coloration, say C. If only 2 or 3 colors are used for the vertices b, d, $\tilde{d}$, $\tilde{b}$ then C extends to a 4-coloring of D as demanded. If all 4 colors are used for b, d, $\tilde{d}$, $\tilde{b}$, say $\alpha, \beta, \gamma, \delta$ then either the $\alpha\gamma$-chain K containing $\tilde{d}$ does not contain b or the $\beta\delta$-chain K' containing $\tilde{b}$ does not contain d (or both); thus we can change C into a 4-coloration C^* of D - a by either exchanging α, γ in K or exchanging β, δ in K'; then C^* uses only 3 colors on b, d, $\tilde{d}$, $\tilde{b}$ and thus can be extended to a 4-coloration of D as demanded. This takes care of Case 2 and proves (D.1).

Proof of (D.2): Assuming the contrary, C would yield a 4-coloration on the triangulation T obtained from D by identifying vertices a and c to vertex v. This would contradict the criticality of T. Q.E.D.

Proof of (D.3): Assuming the contrary, the 12-chain in (D,C) which contains c would not contain a. Then exchanging colors 1,2 in that chain would yield a coloration of D contradicting (D.2). Q.E.D.

4-color-seminar / 13 Proof of coloration theorems (D.4)...(D.7)

(Haken on Shimamoto's construction; October 22, 1971)

Proof of (D.4) and (D.5): Because of (D.1) and (D.2) there is a coloration C of D which colors vertices a, c, b 1, 2, 3 and by (D.3) provides for a 12-chain from a to c. Thus the 34-chain in (D, C) which contains $\tilde{b}$ does not contain b. Exchanging colors 3, 4 in that chain yields a coloration C'. Now either C is as demanded in (D.4) and C' as in (D.5), or the other way.

Proof of (D.6): Let (x, y) be an interior edge in D and denote the corresponding edge in T also by (x, y) (where T is obtained from D by identifying a and c). Then T - (x, y) admits a 4-coloration, say C. The corresponding 4-coloration of D - (x, y) has then same color at a and c, and hence is equivalent to a coloration, say C', which induces one of $\overset{1}{\underset{1}{2\;2}}$, $\overset{1}{\underset{1}{2\;3}}$. Q.E.D.

Proof of (D.7). Case 1: a is of degree 3 in D (left of Fig. on p.12). By (D.5) there is a coloration C of D inducing $\overset{1}{\underset{2}{3\;4}}$ on $\overset{a}{\underset{c}{b\;\tilde{b}}}$. Then, by (D.3), d has color 2. We regard C also as a coloration of D - (a, b). Now in (D - (a, b), C) the 13-chain K containing b does not contain a (since a does not have any neighbor vertex of color 3 in D - (a, b)). Thus exchanging the colors 1, 3 in K yields a coloration inducing $\overset{1}{\underset{2}{1\;4}}$ and thus being equivalent to the demanded coloration. Q.E.D.

Case 2: a is of degree 4 in D (right of Fig. on p. 12; notation as there). As in Case 1 there is a coloration C of D inducing $\overset{1}{\underset{2}{3\;4}}$. Then by (D.3) we have either

Case 2.1: $\tilde{d}$ has color 2, and d has color 4, or
Case 2.2: d has color 2, and $\tilde{d}$ has color 3.

In Case 2.1 we proceed as in Case 1 (i.e., we exchange 1, 3 in the 13-chain K in (D - (a, b), C) which contains b).

In Case 2.2 we have the coloration $\overset{1}{\underset{2}{3\,2\,3\,4}}$ on $\overset{a}{\underset{c}{b\,d\,\tilde{d}\,\tilde{b}}}$.

Then we exchange the colors 3, 4 in the 34-chain which contains b (and which by (D.3) does not contain $\tilde{b}$). This yields a coloration C' of D inducing $\overset{1}{\underset{2}{4\,2\,3\,4}}$. Now we consider the 23-chain K' in (D, C') which contains d and $\tilde{d}$. Then we have either

Case 2.2.1: K' does not contain c, or Case 2.2.2: K' contains c.

4-color-seminar / 14 Proof of (D.7) cont. and of (D.8)

(Haken on Shimamoto's construction; October 22, 1971)

In Case 2.2.1 we exchange colors 2,3 in K' which yields a coloration C"

of D inducing $4\,3\overset{1}{\underset{2}{2}}\,4$. Then we exchange 3,4 in the 34-chain in (D,C") which contains b and d (and by (D.3) not $\tilde{b}$). This yields a coloration C"' of D which induces $3\,4\overset{1}{\underset{2}{2}}\,4$. But then we are in the situation of Case 2.1 (with C"' in place of C) and we proceed as there (exchanging 1,3 in the 13-chain containing b in (D- (a,b), C"')).

In Case 2.2.2 because of 23-chain K' joining $\tilde{d}$ and a, the 14-chain(say K*) in (D- (a,b), C') which contains b does not contain a. Now we exchange 1,4 in K* which yields a coloration equivalent to the demanded one. This finishes the proof of (D.7).

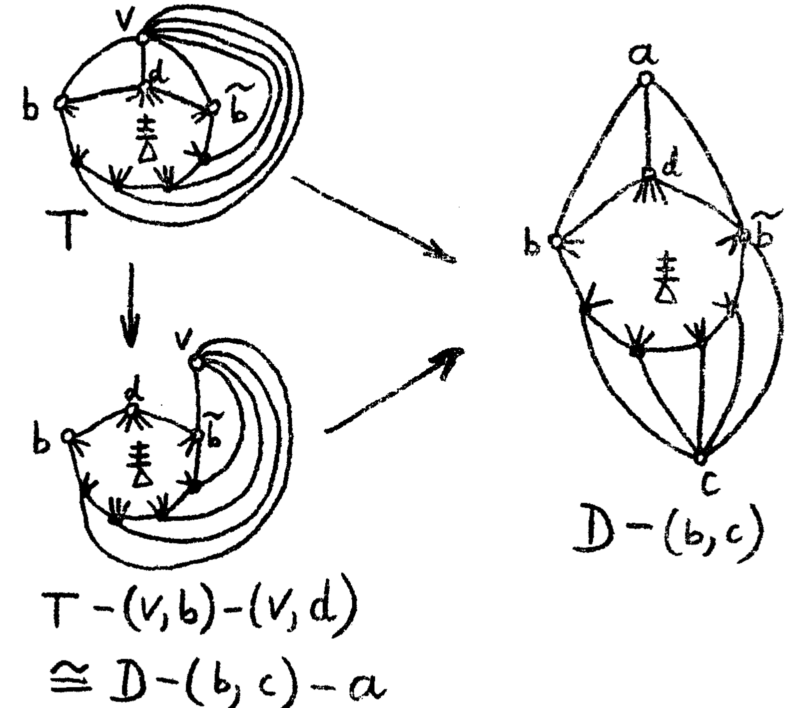
Proof of (D.8): Case 1: a is of degree 3 in D (Figs. pp. 12 and 15). T- (v,b) admits a coloration, say C, which colors both v and b the same (since otherwise T would be not critical). We regard C also as a coloration of T- (v,b) - (v,d) which is "same" as D- (b,c) - a. We regard C also as a coloration of D- (b,c) - a (c corresponding to v). Then C extends to a coloration C' of D- (b,c) (which assigns to a the color different from the colors of b,d,$\tilde{b}$). Now C' (coloring b and c the same) is equivalent to the coloration demanded. Q.E.D.

Case 2: a is of degree 4 in D (Figs. p. 12 right and p.15). Again T- (v,b) admits a coloring C which colors v and b the same. This serves also as a coloring of T- (v,b) - (v,d) - (v,$\tilde{d}$) which is "same" as D- (b,c) - a. Now either Case 2.1: C assigns only 2 or 3 colors to b,d,$\tilde{d}$,$\tilde{b}$, or Case 2.2: C assigns all 4 colors to b,d,$\tilde{d}$,$\tilde{b}$, say $\alpha,\beta,\gamma,\delta$.
In Case 2.1, C extends to a coloration of D- (b,c) (as in Case 1) which is equivalent to the demanded coloration.
In Case 2.2 we identify the vertices v and b of T- (v,b) - (v,d) - (v,$\tilde{d}$) to a vertex, say b' (see Fig.p.15). This yields a graph F which is also colored by C (since both v and b are colored α). Now either the $\alpha\gamma$-chain K in (F,C) which contains $\tilde{d}$ does not contain b' or the $\beta\delta$-chain K' which contains $\tilde{b}$ does not contain d (or both). Then we either exchange α,γ in K or β,δ in K' in order to obtain a coloration C' of F in which only 3 colors are used for b',d,$\tilde{d}$,$\tilde{b}$. The corresponding coloration of D- (b,c) - a extends them to a coloration of D- (b,c) equivalent to the demanded one.

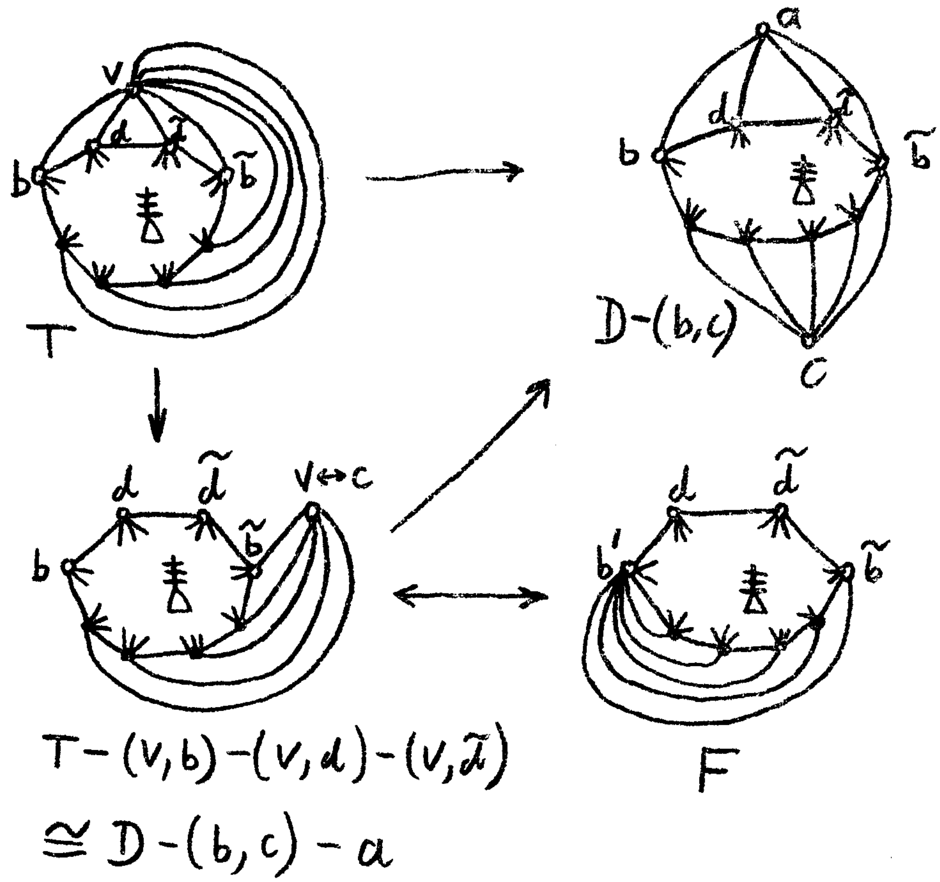
Q.E.D.

 Figures illustrating the proof of (D.8)

(Haken on Shimamoto's constructions October 22, 1971)



Case 1



Case 2

4-color-seminar / 16 Proof of coloration theorem S_2

(Haken on Shimamoto's construction; October 22, 1971)

Proof of $(S_2.1)$: Since S_2 = critical graph minus edges it is 4-colorable.

Proof of $(S_2.2)$: Otherwise $T = S_2 + (a,c) + (a,\tilde{c})$ would be 4-colorable.

Proof of $(S_2.3)$: Otherwise at least one of the 1α-chains $(\alpha = 3,4)$ in (S_2, C) which contains vertex a would not contain c. Then exchanging $1,\alpha$ in that chain would yield a coloration of S_2 contradicting $(S_2.2)$.

Proof of $(S_2.4)$: By $(S_2.1)$ and $(S_2.2)$ S_2 admits a coloration, say C, which induces $\begin{matrix}\beta & 1 & 3\\ 1 & & 2\end{matrix}$ (taken from a coloration of S-graph $T - (a,c)$). By $(S_2.3)$ there is a 13-chain K and a 14-chain K' in (S_2, C) (from a to c). If $\beta = 2$ then C is as demanded. If $\beta = 3$ then we exchange colors 2,3 in the 23-chain which contains b (and because of K' does not contain $\tilde{c}$). If $\beta = 4$ then we exchange 2,4 in the 24-chain that contains b (and because of K does not contain $\tilde{c}$). In each case we obtain a coloration as demanded.

Proof of $(S_2.5)$ and $(S_2.6)$: Let C be a coloration of S_2 as in $(S_2.4)$ inducing $\begin{matrix}2 & 1 & 3\\ 1 & & 2\end{matrix}$ and by $(S_2.3)$ providing 13- and 14-chains from a to c. Then exchanging 2,3 in the 23-chain containing b gives a coloration as demanded for $(S_2.5)$, and exchanging 2,4 in the 24-chain containing b yields a coloration as demanded for $(S_2.6)$. Q.E.D.

Proof of $(S_2.7)$: Let (x,y) be an arbitrary interior edge in S_2 (and denote the corresponding edge in T also by (x,y)). Then $T - (x,y)$ admits a coloration, say C, (which colors $a, c, \tilde{c}$ with 3 different colors). The corresponding coloration of $S_2 - (x,y)$ is then equivalent to the one needed.

Proof of $(S_2.8)$: $T - (a,b)$ admits a coloration C which colors both a and b the same (since T is critical). Then the corresponding coloration of $S_2 - (a,b)$ is equivalent to the demanded coloration, (since C assigns three different colors to $a, c, \tilde{c}$).

Proof of $(S_2.9)$: $T - (b,c)$ admits a coloration C. Then the corresponding coloration on $S_2 - (b,c)$ is equivalent to the demanded one.

Proof of $(S_2.10)$: $T - (c,\tilde{c})$ admits a coloration, say C. Then the corresponding coloration of $S_2 - (c,\tilde{c})$ is equivalent to the demanded coloration.

(Haken on Shimamoto's construction, October 23, 1971)

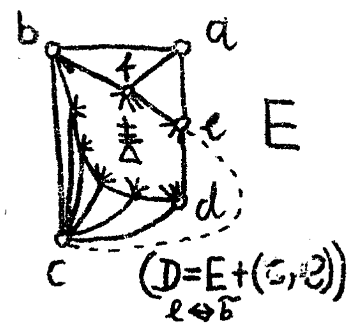
Proof of (E.1): Since E = D - (b̃,c) this follows from (D.1), (see Fig.).

Proof of (E.2): Assuming the contrary, C would yield a corresponding 4-coloration of T (since T can be obtained from E by identifying a and c and the edges (b,a) and (b,c)).

Proof of (E.3): Assuming the contrary, exchanging 1,2 in the 12-chain in (E,C) which contains c would yield a coloration of E contradicting (E.2).

Proof of (E.4) and (E.5): By (D.4) E admits a coloration C inducing 3 1 3 2 α

where α is either 1 or 4. By (E.3) the only interior vertex, say f, of E which neighbors a (see Fig.) is colored 2. Thus the 14-chain in (E,C) containing d does not contain a, and exchanging 1,4 in that chain yields a coloration, say C', of E which induces 3 1 3 2 β, β ≠ α (β = 1 or 4). Then C and C' are the colorations as demanded.



Proof of (E.6) and (E.7): By (E.4) ((by (E.5))) E admits a coloration C inducing 3 1 3 2 1 (inducing 3 1 3 2 4). Exchanging 4,3 in the 43-chain K in (E,C) which contains 2 yields a coloration C' of E inducing 3 1 4 2 1 (3 1 4 2 3) as demanded (since by (E.3) K does not contain b).

Proof of (E.8): By (D.6) E - (interior edge) admits a coloration C inducing 2 1 2 1 α (α = 3 or 4) or a coloration C' inducing 2 1 3 1 β (β = 2 or 4). Now C is equivalent to a coloration inducing 2 1 2 1 3; if β = 2 then C' is equivalent to a coloration inducing 3 1 2 1 3, and if β = 4 then C' is equivalent to a coloration ind. 4 1 2 1 3. In every case E - (interior edge) admits a coloration as demanded.

Proof of (E.9): By (D.8) E - (b,c) admits a coloration C inducing 2 1 3 2 α where either Case 1: α = 1 or Case 2: α = 4. In Case 1, C is as demanded.

(Haken on Shimamoto's construction; October 23, 1971)

In Case 2 we consider the 23-chain K in (E - (b,c),C) which contains e, and we have either Case 2.1: K contains neither a nor c, or Case 2.2: K contains at least one of a,c.

In Case 2.1 we exchange 2,3 in K and obtain a coloration inducing $\begin{smallmatrix}2&1\\&2\\2&4\end{smallmatrix}$.

In Case 2.2 we exchange 1,4 in the 14-chain containing d (which because of K does not contain a) and obtain a coloration inducing $\begin{smallmatrix}2&1\\&3\\2&1\end{smallmatrix}$. This takes care of all cases.

Proof of (E.10): By (D.7) E - (a,b) admits a coloration C inducing $\begin{smallmatrix}1&1\\&3\\2&\alpha\end{smallmatrix}$ where either Case 1: $\alpha = 1$ or Case 2: $\alpha = 4$. In Case 1, C is as demanded. In Case 2, let K be the 23-chain in (E - (a,b),C) which contains e, and we have either Case 2.1: K does not contain c, or Case 2.2: K contains c. In Case 2.1, exchanging 2,3 in K yields a coloration as demanded. In Case 2.2, exchanging 1,4 in the 14-chain containing d (which because of K contains neither a nor b) yields a coloration as demanded. Q.E.D.

Proof of (E.11): Since (c,d) is an interior edge of D = E + (c,e), by (D.6) E - (c,d) admits a coloration inducing $\begin{smallmatrix}2&1\\&2\\1&\alpha\end{smallmatrix}$ or a col. ind. $\begin{smallmatrix}2&1\\&3\\1&\beta\end{smallmatrix}$. Moreover, we must have (in the first case) $\alpha = 1$ and (in the second case) $\beta = 1$ since otherwise we would have a coloration of D coloring a and c the same, in contradiction to (D.2). Thus in every case we have a coloration equivalent to the demanded.

Proof of (E.12): Since (d,e) is an interior edge of D = E + (c,e), by (D.6) E - (d,e) admits a coloration C inducing $\begin{smallmatrix}2&1\\&2\\1&\alpha\end{smallmatrix}$ or a col. ind. $\begin{smallmatrix}2&1\\&3\\1&\beta\end{smallmatrix}$. Moreover, in the first case we have $\alpha = 2$, and in the second case $\beta = 3$ (since otherwise we would have a coloration of D contradicting (D.2)). Thus in every case we have a coloration equivalent to the demanded.

Proof of (E.13): By (E.12) E - (d,e) admits either (Case 1) a coloration C inducing $\begin{smallmatrix}2&1\\&2\\1&2\end{smallmatrix}$ or (Case 2) a coloration C' inducing $\begin{smallmatrix}3&1\\&2\\1&2\end{smallmatrix}$.

In Case 1, let α be the color of vertex f (Fig.p.17) ($\alpha = 3$ or 4); let β be 3 or 4 but $\neq \alpha$. Exchanging 1,β in the 1β-chain in (E - (d,e),C)

(Haken on Shimamoto's construction; October 23, 1971)

which contains c (but does not contain a) yields a coloration equivalent to one

inducing $\begin{smallmatrix} 3 & 1 \\ & 3 \\ 2 & 3 \end{smallmatrix}$ as demanded.

In Case 2 let K be the 14-chain in (E - (d,e), C') which contains c. Then either Case 2.1: K contains a, or Case 2.2: K does not contain a.
In Case 2.1 we exchange 2,3 in the 23-chain in (E - (d,e), C') that contains b (and because of K neither contains d nor e) and obtain a coloration C^*

inducing $\begin{smallmatrix} 2 & 1 \\ & 2 \\ 1 & 2 \end{smallmatrix}$. Then we apply the procedure described in Case 1 with C^* instead of C, which takes care of Case 2.1.

In Case 2.2 we exchange 1,4 in K and obtain a coloration equivalent to one as demanded. This proves (E.13).

Proof of (E.14): T - (v, $\tilde{b}$) (see Fig. on p.7) admits a 4-coloration in which (since T is critical) v and $\tilde{b}$ are colored the same; an equivalent coloration, say C, of T - (v, $\tilde{b}$) colors v and $\tilde{b}$ both 1. This yields a coloration, also called C, of the graph D - (a, $\tilde{b}$) - (c, $\tilde{b}$) (as obtained from T - (v, $\tilde{b}$) by splitting vertex v into a and c) which assigns color 1 to a, $\tilde{b}$, and c. But D - (a, $\tilde{b}$) - (c, $\tilde{b}$) is same as E - (a,e) (where $\tilde{b}$ is renamed "e"). Thus C is equivalent to a coloration C' of E - (e,a) inducing $\begin{smallmatrix} 2 & 1 \\ & 1 \\ 1 & \alpha \end{smallmatrix}$,

where α is either 2 or 3. In each case C' is equivalent to a coloration as demanded.

Proof of (E.15): Exchanging the roles of b and $\tilde{b}$ in the proof of (D.7) we conclude that D - (a, $\tilde{b}$) admits a coloration C inducing $\begin{smallmatrix} & 1 & \\ 3 & & 1 \\ & 2 & \end{smallmatrix}$. Since

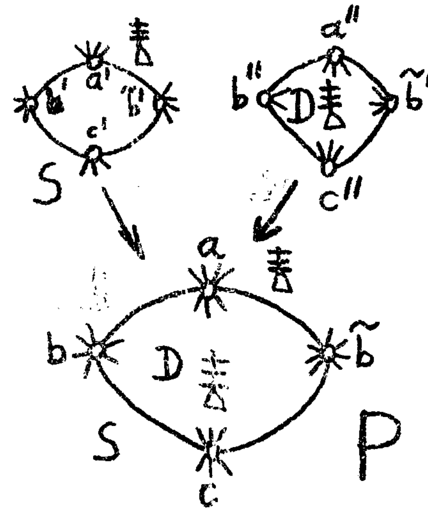
E - (e,a) = D - (a, $\tilde{b}$) - ($\tilde{b}$, c) ($\tilde{b}$ being named "e" in E)

this is also a coloration of E - (e,a) inducing $\begin{smallmatrix} 3 & 1 \\ & 1 \\ 2 & \alpha \end{smallmatrix}$ where α is either 3 or 4.

In each case C is equivalent to a coloration as demanded.

4-color-seminar / 20 Statement of the Construction theorems

(Haken on Shimamoto's construction; October 23, 1971)



Construction Theorem I: Let S be an S-graph with boundary circuit $\begin{smallmatrix} & a' & \\ b' & & \tilde{b}' \\ & c' & \end{smallmatrix}$ (special vertices a' and c').

Let D be a D-graph with boundary circuit $\begin{smallmatrix} & a'' & \\ b'' & & \tilde{b}'' \\ & c'' & \end{smallmatrix}$ (top vertex a'', bottom vertex c'').

Let graph P be obtained from S and D by identifying their boundary circuits so that a', a'' are identified to a, b', b'' to b, etc. (see Fig.).

Then P is a critical triangulation.

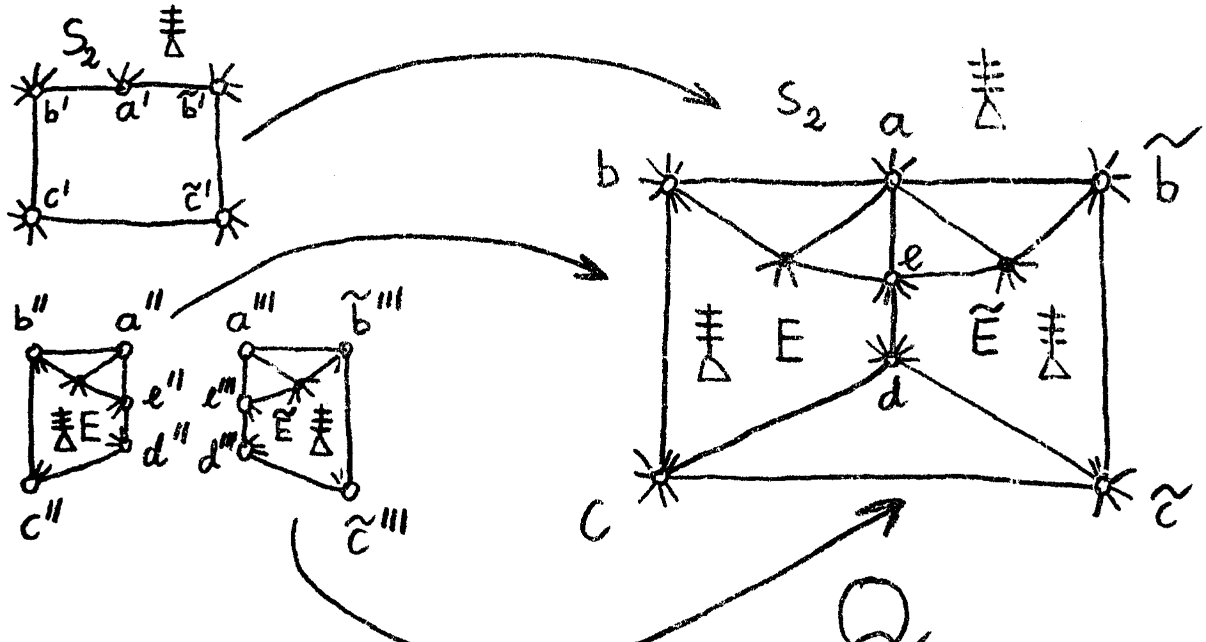
Construction Theorem II: Let S_2 be an S_2-graph with boundary circuit $\begin{smallmatrix} b' & a' & \tilde{b}' \\ c' & & \tilde{c}' \end{smallmatrix}$ (top vertex a', bottom vertices c' and $\tilde{c}'$).

Let E be an E-graph with boundary circuit $\begin{smallmatrix} b'' & a'' \\ & e'' \\ c'' & d'' \end{smallmatrix}$ (top vertex a'', bottom c'').

Let $\tilde{E}$ be another E-graph with boundary circuit $\begin{smallmatrix} a''' & \tilde{b}''' \\ e''' & \\ d''' & \tilde{c}''' \end{smallmatrix}$ (top vertex a''', bottom vertex $\tilde{c}'''$).

Let graph Q be obtained from S_2, E, and $\tilde{E}$ by (partially) identifying their boundary circuits so that a', a'', and a''' are identified to a, and b', b'' are identified to b, and $\tilde{b}'$, $\tilde{b}'''$ are identified to $\tilde{b}$, etc. (see Fig. below).

Then Q is a critical triangulation.



(Haken on Shimamoto's construction; October 23, 1971)

Proof of Construction Theorem I: P is a triangulation by construction.

P is not 4-colorable since a 4-coloration C of P would either contradict (S.2) or contradict (D.2).

It remains to be proved that P − (arbitrary edge) is 4-colorable.
We prove this by means of the coloration theorems S and D (p.8).

(I.1) P − (interior edge of S) admits a 4-coloration which induces on $\begin{array}{ccc} & a & \\ b & & \tilde{b} \\ & c & \end{array}$

either $\begin{array}{ccc} & 1 & \\ 3 & & 3 \\ & 2 & \end{array}$ by $\begin{cases} \text{(S.6), Case 1} \\ \text{(D.4)} \end{cases}$, or $\begin{array}{ccc} & 1 & \\ 3 & & 4 \\ & 2 & \end{array}$ by $\begin{cases} \text{(S.6), Case 2} \\ \text{(D.5)} \end{cases}$.

(I.2) P − (interior edge of D) admits a 4-coloration which induces

either $\begin{array}{ccc} & 1 & \\ 2 & & 2 \\ & 1 & \end{array}$ by $\begin{cases} \text{(D.6), Case 1} \\ \text{(S.4)} \end{cases}$, or $\begin{array}{ccc} & 1 & \\ 2 & & 3 \\ & 2 & \end{array}$ by $\begin{cases} \text{(D.6), Case 2} \\ \text{(S.5)} \end{cases}$.

(I.3) P − (a,b) admits a 4-coloration which induces $\begin{array}{ccc} & 1 & \\ 1 & & 3 \\ & 2 & \end{array}$ by $\begin{cases} \text{(S.7)} \\ \text{(D.7)} \end{cases}$.

(I.$\tilde{3}$) P − (a,$\tilde{b}$) admits a 4-coloration. A proof of this is obtained by exchanging the roles of b and $\tilde{b}$ in (I.3) (and in the proofs of (S.7) and (D.7) which are used for proving (I.3)).

(I.4) P − (b,c) admits a 4-coloration which induces $\begin{array}{ccc} & 1 & \\ 2 & & 3 \\ & 2 & \end{array}$ by $\begin{cases} \text{(S.8)} \; (1 \leftrightarrow 2) \\ \text{(D.8)} \end{cases}$.

(I.$\tilde{4}$) P − ($\tilde{b}$,c) admits a 4-coloration. A proof of this is obtained by exchanging the roles of b and $\tilde{b}$ in (I.4).

(I.1) . . . (I.$\tilde{4}$) imply that P − (arbitrary edge) is 4-colorable. Q.E.D.

Proof of Construction Theorem II: Q is a triangulation by construction.

Q is not 4-colorable since a 4-coloration C of Q would either contradict (S_2.2) or contradict (E.2).

It remains to be proved that Q − (arbitrary edge) is 4-colorable.
We prove this by means of the coloration theorems S_2 (p.9) and E (p.10).

(II.1) Q − (interior edge of S_2) admits a 4-coloration inducing on $\begin{array}{ccc} b & a & \tilde{b} \\ & e & \\ c & d & \tilde{c} \end{array}$

either $\begin{array}{ccc} 4 & 1 & 4 \\ & 4 & \\ 2 & 1 & 3 \end{array}$ by $\begin{cases} (S_2.7)\text{, Case 1} \\ \text{(E.4)} (3 \leftrightarrow 4) \\ \text{(E.4)} (2 \to 3 \to 4) \end{cases}$, or $\begin{array}{ccc} 3 & 1 & 2 \\ & 4 & \\ 2 & 1 & 3 \end{array}$ by $\begin{cases} (S_2.7)\text{, Case 2} \\ \text{(E.6) (on } E) \\ \text{(E.6)} (2 \leftrightarrow 3) \text{ (on } \tilde{E}) \end{cases}$

(indicating the permutation of colors in (E.4))

or (cont. next p.)

(Haken on Shimamoto's construction; October 24, 1971)

or $\begin{matrix}3&1&4\\&4&\\2&1&3\end{matrix}$ by $\left\{\begin{array}{l}(S_2.7),\text{ Case }3\\(E.6)\\(E.4)(2\to3\to4)\end{array}\right.$, or $\begin{matrix}4&1&2\\&4&\\2&1&3\end{matrix}$ by $\left\{\begin{array}{l}(S_2.7),\text{ Case }4\\(E.4)(3\leftrightarrow4)\\(E.6)(2\leftrightarrow3)\end{array}\right.$.

(II.2) Q - (interior edge of E) admits a 4-coloration which induces

either $\begin{matrix}2&1&2\\&2&\\&3&\\1&&4\end{matrix}$ by $\left\{\begin{array}{l}(S_2.5)(3\to2\to4)\\(E.8),\text{ Case }1\\(E.5)(3\to2\to4)\end{array}\right.$, or $\begin{matrix}3&1&3\\&2&\\&3&\\1&&4\end{matrix}$ by $\left\{\begin{array}{l}(S_2.5)(2\leftrightarrow4)\\(E.8),\text{ Case }2\\(E.7)(2\leftrightarrow4)\end{array}\right.$,

or $\begin{matrix}4&1&3\\&2&\\&3&\\1&&4\end{matrix}$ by $\left\{\begin{array}{l}(S_2.4)(2\leftrightarrow4)\\(E.8),\text{ Case }3\\(E.7)(2\leftrightarrow4)\end{array}\right.$.

$(II.\tilde{2})$ Q - (interior edge of $\tilde{E}$) admits a 4-coloration. A proof of this is obtained by exchanging the roles of E and $\tilde{E}$, of b and $\tilde{b}$, and of c and $\tilde{c}$ in (II.3) (and in the proofs of the coloration theorems needed there).

(II.3) Q - (c,d) admits a 4-coloration which induces

either $\begin{matrix}2&1&2\\&2&\\&1&\\1&&3\end{matrix}$ by $\left\{\begin{array}{l}(S_2.5)(2\leftrightarrow3)\\(E.11),\text{ Case }1\\(E.4)(2\leftrightarrow3)\end{array}\right.$, or $\begin{matrix}3&1&2\\&2&\\&1&\\1&&3\end{matrix}$ by $\left\{\begin{array}{l}(S_2.4)(2\leftrightarrow3)\\(E.11),\text{ Case }2\\(E.4)(2\leftrightarrow3)\end{array}\right.$.

$(II.\tilde{3})$ Q - ($\tilde{c}$,d) admits a 4-coloration. A proof of this is obtained by exchanging the roles of E and $\tilde{E}$, of b and $\tilde{b}$, and of c and $\tilde{c}$ in (II.3) (and in the proofs of the coloration theorems needed there).

(II.4) Q - (c,$\tilde{c}$) admits a 4-coloration which induces

either $\begin{matrix}3&1&3\\&4&\\&1&\\2&&2\end{matrix}$ by $\left\{\begin{array}{l}(S_2.10),\text{ Case }1\\(E.6)\\(E.6)\end{array}\right.$, or $\begin{matrix}3&1&4\\&4&\\&1&\\2&&2\end{matrix}$ by $\left\{\begin{array}{l}(S_2.10),\text{ Case }2\\(E.6)\\(E.4)(3\leftrightarrow4)\end{array}\right.$.

(II.5) Q - (b,c) admits a 4-coloration which induces

either $\begin{matrix}2&1&2\\&2&\\&4&\\2&&3\end{matrix}$ by $\left\{\begin{array}{l}(S_2.9),\text{ Case }1\\(E.9),\text{ Case }1\\(E.5)(2\leftrightarrow3)\end{array}\right.$, or $\begin{matrix}2&1&2\\&3&\\&1&\\2&&4\end{matrix}$ by $\left\{\begin{array}{l}(S_2.9),\text{ Case }1,(3\leftrightarrow4)\\(E.9),\text{ Case }2\\(E.6)(2\to4\to3)\end{array}\right.$,

or $\begin{matrix}2&1&4\\&2&\\&4&\\2&&3\end{matrix}$ by $\left\{\begin{array}{l}(S_2.9),\text{ Case }2\\(E.9),\text{ Case }1\\(E.7)(2\to3\to4)\end{array}\right.$, or $\begin{matrix}2&1&3\\&3&\\&1&\\2&&4\end{matrix}$ by $\left\{\begin{array}{l}(S_2.9),\text{ Case }2,(3\leftrightarrow4)\\(E.9),\text{ Case }2\\(E.4)(2\leftrightarrow4)\end{array}\right.$.

$(II.\tilde{5})$ Q - ($\tilde{b}$,$\tilde{c}$) admits a 4-coloration. A proof of this is obtained by exchanging the roles of E and $\tilde{E}$, of b and $\tilde{b}$, and of c and $\tilde{c}$ in (II.5).

(Haken on Shimamoto's construction; October 24, 1971)

(II.6) $Q=(a,b)$ admits a 4-coloration which induces

either $\begin{matrix}1&1&2\\&2&\\2&4&3\end{matrix}$ by $\begin{cases}(S_2.8),\text{ Case }1\\(E.10),\text{ Case }1,\\(E.5)(2\leftrightarrow 3)\end{cases}$ or $\begin{matrix}1&1&2\\&3&\\2&1&4\end{matrix}$ by $\begin{cases}(S_2.8),\text{ Case }1,(3\leftrightarrow 4)\\(E.10),\text{ Case }2\\(E.6)(3\to 2\to 4)\end{cases}$,

or $\begin{matrix}1&1&4\\&2&\\2&4&3\end{matrix}$ by $\begin{cases}(S_2.8),\text{ Case }2\\(E.10),\text{ Case }1\\(E.7)(2\to 3\to 4)\end{cases}$, or $\begin{matrix}1&1&3\\&3&\\2&1&4\end{matrix}$ by $\begin{cases}(S_2.8),\text{ Case }2,(3\leftrightarrow 4)\\(E.10),\text{ Case }2\\(E.4)(2\leftrightarrow 4)\end{cases}$.

(II.$\tilde{6}$) $Q=(a,\tilde{b})$ admits a 4-coloration. A proof of this is obtained by exchanging the roles of E and $\tilde{E}$, of b and $\tilde{b}$, and of c and $\tilde{c}$ in (II.6).

(II.7) $Q=(a,e)$ admits a 4-coloration which induces

either $\begin{matrix}2&1&2\\&1&\\1&2&3\end{matrix}$ by $\begin{cases}(S_2.5)(2\leftrightarrow 3)\\(E.14),\text{ Case }1\\(E.15),\text{ Case }1,\ (2\leftrightarrow 3)\end{cases}$, or $\begin{matrix}2&1&4\\&1&\\1&2&3\end{matrix}$ by $\begin{cases}(S_2.6)(2\to 3\to 4)\\(E.14),\text{ Case }1\\(E.15),\text{ Case }2,(2\leftrightarrow 3)\end{cases}$,

or $\begin{matrix}3&1&2\\&1&\\1&2&3\end{matrix}$ by $\begin{cases}(S_2.4)(2\leftrightarrow 3)\\(E.14),\text{ Case }2\\(E.15),\text{ Case }1,(2\leftrightarrow 3)\end{cases}$, or $\begin{matrix}3&1&4\\&1&\\1&2&3\end{matrix}$ by $\begin{cases}(S_2.4)(2\to 3\to 4)\\(E.14),\text{ Case }2\\(E.15),\text{ Case }2,(2\leftrightarrow 3)\end{cases}$.

(II.8) $Q=(d,e)$ admits a 4-coloration which induces

either $\begin{matrix}2&1&2\\&2&\\1&2&3\end{matrix}$ by $\begin{cases}(S_2.5)(2\leftrightarrow 3)\\(E.12),\text{ Case }1\\(E.13),\text{ Case }1,(2\leftrightarrow 3)\end{cases}$, or $\begin{matrix}2&1&4\\&2&\\1&2&3\end{matrix}$ by $\begin{cases}(S_2.6)(2\to 3\to 4)\\(E.12),\text{ Case }1\\(E.13),\text{ Case }2,(2\leftrightarrow 3)\end{cases}$,

or $\begin{matrix}3&1&2\\&2&\\1&2&3\end{matrix}$ by $\begin{cases}(S_2.4)(2\leftrightarrow 3)\\(E.12),\text{ Case }2\\(E.13),\text{ Case }1,(2\leftrightarrow 3)\end{cases}$, or $\begin{matrix}3&1&4\\&2&\\1&2&3\end{matrix}$ by $\begin{cases}(S_2.4)(2\to 3\to 4)\\(E.12),\text{ Case }2\\(E.13),\text{ Case }2,(2\leftrightarrow 3)\end{cases}$.

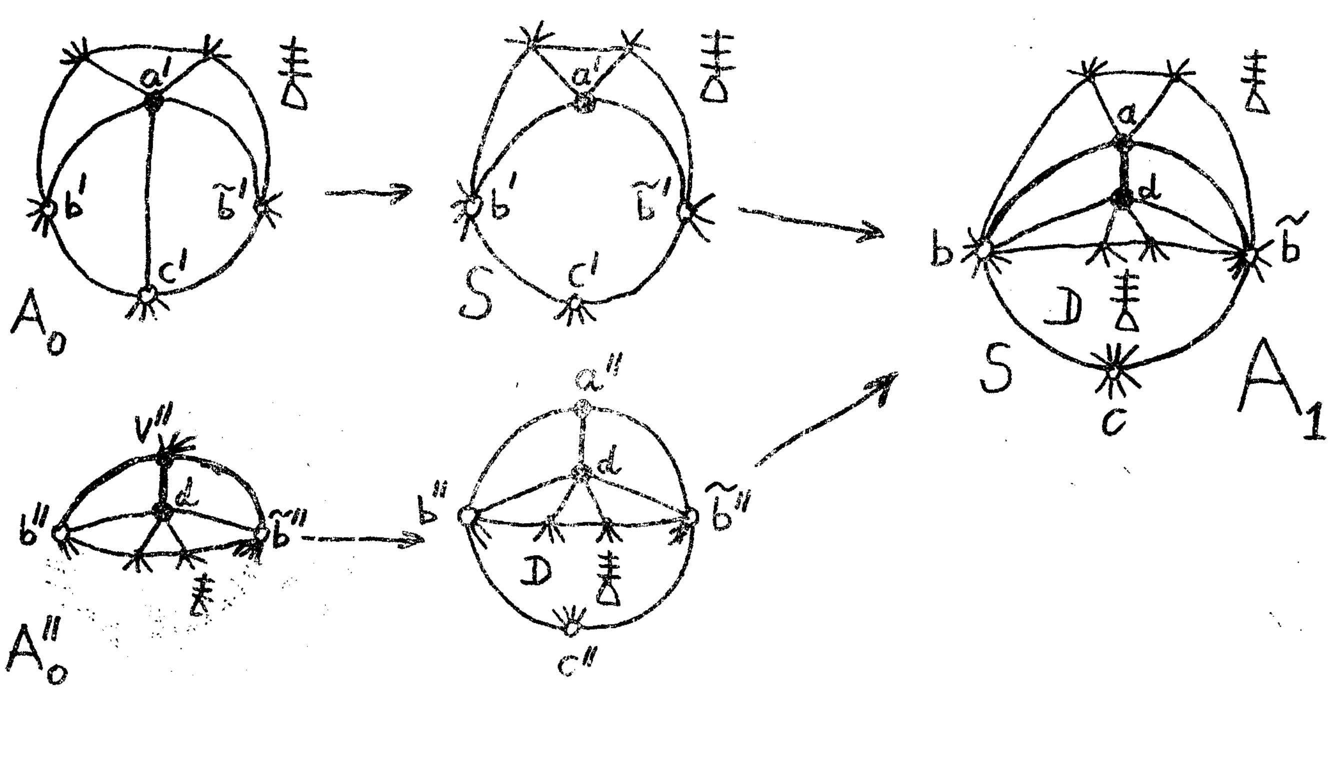
(II.1) . . . (II.8) imply that $Q=$ (arbitrary edge) is 4-colorable. Q.E.D.

(Haken on Shimamoto's construction, October 24, 1971)

Proof of Theorem A_1: The critical triangulation A_1 (as required for a proof of Theorem A_1, see p.3) is derived from the critical triangulation A_0 using Construction Theorem I (see p.20) as follows. (See Fig. below.) Let a' be a vertex of degree 5 in A_0, and let c' be a vertex of A_0 neighboring a'. Then let graph S be equal to $A_0 - (a', c')$ and denote the two other vertices on the boundary circuit of S by b' and $\tilde{b}'$. Now S is an S-graph by def. (see p.7).

Let d be a vertex of degree 5 in another copy, say A_0'', of A_0, and let b'', v'', $\tilde{b}''$ be three neighboring vertices of d in A_0'' lying around d in that order. Then let D be the graph obtained from A_0'' by splitting vertex v'' into vertices a'' and c'' where the cut is done along the edges (b'', v'') and (v'', $\tilde{b}''$) so that a'' is neighboring d in D. Now D is a D-graph by def. (see p.7) (and a'' is of degree 3 in D having neighbor vertices b'', d, $\tilde{b}''$).

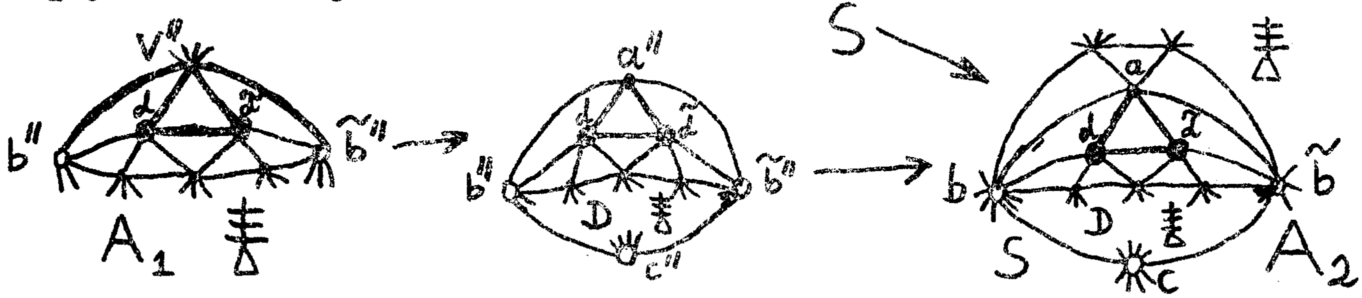
Finally A_1 is obtained from S and D by identifying their boundary circuits in such a way that a' and a'' are identified to a vertex a, and that b', b'' are identified to b, etc. Now A_1 is a critical triangulation by Construction Theorem I. Moreover, A_1 contains the 55-edge (a, d). Thus A_1 has all the required properties and Theorem A_1 is proved.



(Haken on Shimamoto's construction; October 24, 1971)

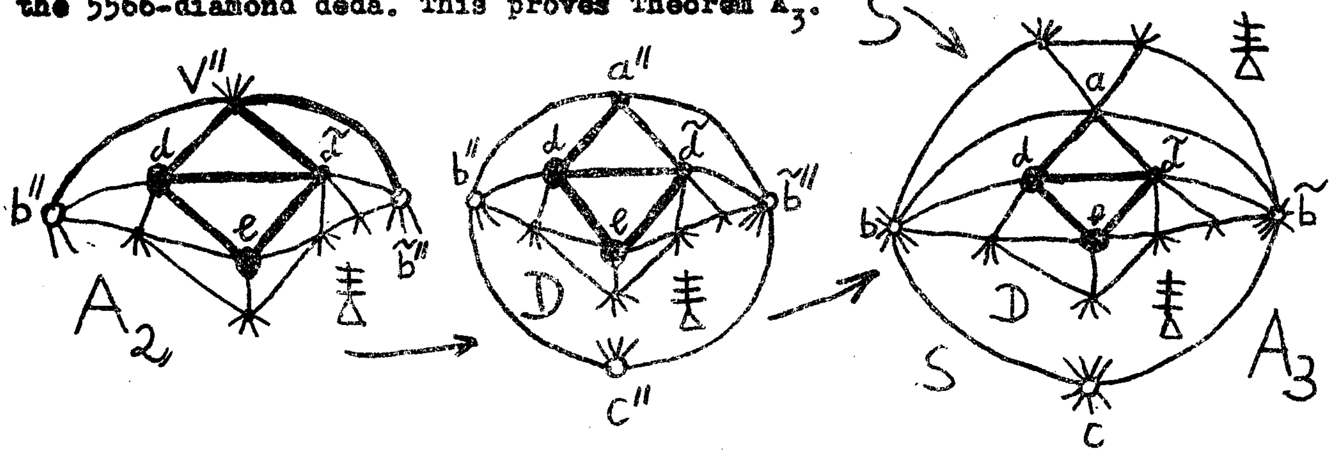
Proof of Theorem A_2: Let S-graph S be as in the proof of Theorem A_1. Let $(d,\tilde{d})$ be a 55-edge in the critical triangulation A_1 and let v" be a vertex of A_1 neighboring both d and $\tilde{d}$. (See Fig. below.) Denote by b" (by $\tilde{b}$") the vertex of A_1 which neighbors both v" and d (and $\tilde{d}$) but is different from $\tilde{d}$ (from d). Then let D be the graph obtained from A_1 by splitting vertex v" into two vertices a" and c" where the cut is done along the edges (b",v") and (v",$\tilde{b}$") so that a" is neighboring d and $\tilde{d}$ in D. Now D is a D-graph by def. (and a" is of degree 4 in D).

Then A_2 is obtained from S and D by identifying their boundary circuits as described in the hypothesis of Construction Theorem I. By that A_2 is a critical triangulation and, moreover, contains the 556-triangle $d\tilde{d}a$. Q.E.D.



Proof of Theorem A_3: Let S-graph S be as in the proof of Theorem A_1. Let $de\tilde{d}$ be a 556-triangle in the critical triangulation A_2 and let v" be the vertex of A_2 which neighbors both d and $\tilde{d}$ but is different from e. (See Fig. below.) Then let D be the graph obtained from A_2 by splitting v" into a" and c'' precisely as described in the proof of Theorem A_2.

Then A_3 is obtained from S and D by identifying according to Construction Theorem I. By this A_3 is a critical triangulation and, moreover, contains the 5566-diamond $de\tilde{d}a$. This proves Theorem A_3.



(Haken on Shimamoto's construction, October 24, 1971)

Proof of Theorem A: The critical triangulation A (as required for a proof of Theorem A, see p.3) is derived from the critical triangulations A_o and A_3 using Construction Theorem II (p.20) as follows. (See Fig. p.27.)

Let a' be a vertex of degree 5 in A_o, and let c', $\tilde{c}'$ be two vertices of A_o which are neighboring a' and each other. Then let graph S_2 be equal to $A_o - (a',c') - (a',\tilde{c}')$ and denote the two vertices other than a', c', $\tilde{c}'$ on the boundary circuit of S_2 by b' (neighboring c') and $\tilde{b}'$ (neighboring $\tilde{c}'$). Now S_2 is an S_2-graph by def. (see p.7).

Let $\tilde{b}''$fgh be a 5566-diamond in A_3, and let v'' be the vertex of A_3 which is neighboring both $\tilde{b}''$ and h but is different from g. Denote by b'' the vertex of A_3 which is neighboring both v'' and h but is different from $\tilde{b}''$. Finally denote by d'' the vertex of A_3 which is neighboring $\tilde{b}''$ and f but is different from g. Then let D be the graph obtained from A_3 by splitting vertex v'' into two vertices a'' and c'' where the cut is done along the edges (b'',v'') and (v'',$\tilde{b}''$) so that a'' is neighboring h in D. Now D is a D-graph by def. (see p.7) (and a'' is of degree 3 in D).
Let graph E be obtained from D by deleting edge ($\tilde{b}''$,c'') and renaming vertex $\tilde{b}''$ to e''. Then E is an E-graph by def.
Let graph $\tilde{E}$ be another copy of graph E, (e.g. a mirror image of E) denoting the vertices of $\tilde{E}$ corresponding to a'',e'',d'' by a''', e''', d''', and the vertices corresponding to f,g,h by $\tilde{f}$, $\tilde{g}$, $\tilde{h}$, and the vertices corresponding to b'',c'' by $\tilde{b}'''$, $\tilde{c}'''$.

Finally let graph A be obtained from S_2, E, and $\tilde{E}$ by (partially) identifying their boundary circuits as described in the hypothesis of Construction Theorem II. By this A is a critical triangulation and, moreover, contains the 85666665-horseshoe efgha$\tilde{h}\tilde{g}\tilde{f}$. Thus A has all the required properties and Theorem A is proved.

 Fig. Illustrating construction of A

(Haken on Shimamoto's construction; October 24, 1971)